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DEVELOPMENT OF ROTATION SAMPLE DESIGNS FOR THE ESTIMATION OF CROP ACREAGES

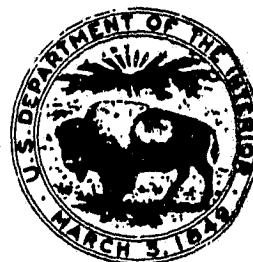
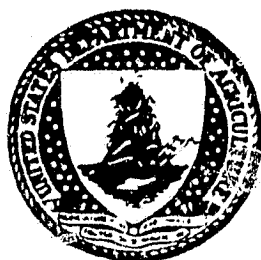
T. G. Lycthuan-Lee

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16. Abstract This paper presents the development of "rotation sample designs" that are useful in crop survey tasks of the Foreign Commodity Production Forecasting project of the AgRISTARS program. The idea behind the use of rotation sample designs is that the variation of the crop acreages of a particular sample unit from year to year is usually less than the variation of crop acreages between units within a particular year. The estimation theory is based on an additive mixed analysis of variance model with years as fixed effects, α_t , and sample units as a variable factor. The rotation patterns are decided upon according to (1) the number of sample units in the design each year, (2) the number of units retained in the following years, and (3) the number of years to complete the rotation pattern. Different analytic formulae for the variances of α_t and the variance reduction ratios are presented. An optimal design is established based on numerical comparisons in using a complete survey of the rotation patterns.					
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DEVELOPMENT OF ROTATION SAMPLE DESIGNS FOR THE
ESTIMATION OF CROP ACREAGES


Job Order 72-422


This report describes Sampling and Aggregation activities of the
Foreign Commodity Production Forecasting project of the AgRISTARS program.

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PREFACE

The Agriculture and Resources Inventory Surveys Through Aerospace Remote Sensing is a multiyear program of research, development, evaluation, and application of aerospace remote sensing for agricultural resources, which began in fiscal year 1980. This program is a cooperative effort of the U.S. Department of Agriculture, the National Aeronautics and Space Administration, the National Oceanic and Atmospheric Administration (U.S. Department of Commerce), the Agency for International Development (U.S. Department of State), and the U.S. Department of the Interior.

The work which is the subject of this document was performed by the Earth Resources Applications Division, Space and Life Sciences Directorate, Lyndon B. Johnson Space Center, National Aeronautics and Space Administration and Lockheed Engineering and Management Services Company, Inc. The task performed by Lockheed Engineering and Management Services Company, Inc., were accomplished under Contract NAS 9-15800. A major portion of this task was performed at Texas A&M University in partial fulfillment of requirements for a doctorate degree in statistics under professor H. O. Hartley (supported on contract NAS9-13894).

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1. INTRODUCTION

1.1 THE AGRICULTURE AND RESOURCES INVENTORY THROUGH AEROSPACE REMOTE SENSING PROGRAM (AgRISTARS)

1.1.1 DESCRIPTION AND OBJECTIVE OF THE AgRISTARS PROGRAM

The AgRISTARS program officially started in fiscal year 1980 after a successful completion of the Large Area Crop Inventory Experiment (LACIE). The LACIE was a project to test the technology that would be used in a satellite-aided crop production estimating system utilizing remote sensing. Instruments onboard Earth-observing satellites (i.e., Landsat) were used to obtain information about the Earth by scanning its surface from the orbiting satellite.

Over the past two decades, many developments led to the technologies used in LACIE. Some notable developments were:

- (1) Multispectral scanners capable of scanning the Earth's surface and producing a quantitative radiometric map at visible, near-infrared, and thermal infrared wave-lengths;
- (2) pattern recognition techniques permitting agriculture crops to be identified on the basis of differences in spectral response during the year;
- (3) high-speed digital computers;
- (4) the 1972 launch by the National Aeronautics and Space Administration (NASA) of the first of the Land Satellite (Landsat) series of polar-orbiting satellites, making it possible to monitor each point on the globe every 18 days with multispectral scanners;
- (5) the development of a global weather-reporting network by the World Meteorological Organization (WMO);
- (6) the development of models capable of relating weather to crop yields (refs. 1 and 2); and
- (7) the development of many custom-made statistical methodologies.

The LACIE completion proved that Earth-observing satellites used to gather data on agricultural and other resources can provide foreign agricultural production information with accuracy and timeliness.

The successes of LACIE have stimulated an effort to "develop and evaluate aerospace remote sensing technology for other major commodities and global crop regions" (ref. 1, pp. 13-15) through its new program named AgRISTARS.

"The specific objectives of AgRISTARS include development, testing, and evaluation of procedures for adapting space remote sensing technology to improve the U.S. Department of Agriculture (USDA) capabilities to provide early warning and timely assessment of changes in crop conditions; to provide more objective and reliable crop production forecasts; to assist in inventory and assessment of land, water, and other renewable resources; and to develop a cost base to permit the USDA to assess feasibility of integrating space remote sensing technology with existing data systems." (ref. 1, pp. 13-15)

1.1.2 GENERAL APPROACH OF THE PROJECT

The project develops technology for use in processing Landsat data primarily to inventory the amount and geographic distribution of crop acreage available for harvest. It uses meteorological data in agricultural-meteorological models to forecast yield per harvested acre (production).

"This general approach requires that each crop region be stratified with relatively homogeneous subregions and randomly selected samples of Landsat data be machine processed to identify and measure the areal extent of the crop types of interest. Yield models developed over subregions by the Yield Model Development Project in AgRISTARS will be exercised to forecast yield. The estimates of area and yield will be combined to estimate area, yield, and production at the regional levels." (ref. 2, p. I.4)

Symbolically, the production estimation approach is presented as follows. The crop region is stratified into L strata: 1, 2, ..., h, ..., L; and

$$(\text{CROP ACRFAGE})_{\text{stratum}} \times (\text{CROP YIELD})_{\text{stratum}} = (\text{CROP PRODUCTION})_{\text{stratum}};$$

$$(\text{CROP PRODUCTION})_{\text{region}} = \sum_{h=1}^L (\text{CROP PRODUCTION})_{\text{stratum } h}$$

or $(\text{CROP PRODUCTION})_{\text{region}} = (\text{CROP ACREAGE})_{\text{region}} \times (\text{CROP YIELD})_{\text{region}}.$

1.1.3 SCOPE OF THE PROJECT

The project involves many different crops such as wheat, barley, rice, corn, soybeans, cotton, sorghum, and sunflowers, over many selected regions within the United States, Canada, India, U.S.S.R., China, Australia, Argentina, and Brazil.

1.1.4 SAMPLE DESIGN USED IN THE PROJECT

1.1.4.1 Description of the Basic Sample Design

The objective of a sample design for the project is to estimate the crop acreage for each of J crops in a particular region of a specific country. The basic sample design of the project is the stratified simple random sample design. At least, this basic design is used in a 'Group I' situation*. Let A_j be the crop acreage for crop j , $j = 1, 2, \dots, J$ in a designated crop region of a specific country.

The crop region is divided into L subregions called strata which are relatively homogeneous in terms of the crop density (hence, the crop acreage, yield, and production).

The sampling units are the acreages of the 5-by-6 nautical mile area. These units are called 'segments'.

For $h = 1, 2, \dots, L$, let N_h be the total number of segments made up of the h^{th} stratum and n_h be the number of segments randomly selected by the simple random sampling (SRS) from the N_h segments of stratum h .

Let A_{hj} be the crop acreage for crop j in the stratum h of the crop region under consideration and A_{hji} be the total crop acreage for crop j in the segment i randomly selected in the stratum h .

It is supposed to have the mean acreage per segment as

$$\bar{A}_{hj} = \sum_{i=1}^{N_h} A_{hji} / N_h \quad (1)$$

*The 'Group I' situation is one in which the stratum sample size n_h is greater than 1, $h = 1, 2, \dots, L$.

but the SRS plan gives only n_h segments, i.e., n_h crop acreages A_{hji} , $i = 1, 2, \dots, n_h$. However, the SRS design gives an unbiased estimate of \bar{A}_{hj} , which is

$$\hat{\bar{A}}_{hj} = \sum_{i=1}^{n_h} A_{hji}/n_h \equiv \bar{a}_{hj}. \quad (2)$$

which is the mean acreage of crop j per segment in the n_h segments sampled in stratum h .

The stratified unbiased estimator of A_j is

$$\hat{A}_j = \sum_{h=1}^L N_h \hat{\bar{A}}_{hj}. \quad (3)$$

The unbiased estimator \hat{A}_j has variance

$$v(\hat{A}_j) = \sum_{h=1}^L N_h^2 (1 - n_h/N_h) (s_{hj}^2/n_h) \quad (4)$$

where s_{hj}^2 is the variance within the h^{th} stratum of the segment acreage of crop j ; i.e.,

$$s_{hj}^2 = \sum_{i=1}^{N_h} (A_{hji} - \bar{A}_{hj})^2 / (N_h - 1) \quad (5)$$

where

$$\bar{A}_{hj} = \sum_{i=1}^{N_h} A_{hji} / N_h \quad (6)$$

the mean acreage of crop j per segment in the N_h segments of stratum h .

Then, the estimate of the production of crop j in the region will be

$$(\text{Production of crop } j)_{\text{region}} = \hat{A}_j \times \hat{\bar{Y}}_j \quad (7)$$

where

$\hat{\bar{Y}}_j$ is the estimate of the average yield of crop j in the region under consideration.

1.1.4.2 Flow Chart of a General Approach of Estimation

A general approach used in estimation procedures can be seen in the flow chart shown in figure 1-1.

1.1.4.3 Some Drawbacks of the Stratified Simple Random Sample Design

One important drawback of the stratified simple random sample design is that it does not utilize the fact of the existence of a significant positive correlation between the crop segment acreages observed in the current and previous years in order to yield more accurate estimates.

Another shortcoming is that the stratified simple random sample design cannot overcome the problem of missing data. The stratified estimate

$$\hat{A}_j = \sum_{h=1}^L N_h \hat{\bar{A}}_{hj}.$$

which is shown as equation 3, will be badly effected when, for any reason, one stratum acreage estimate cannot be obtained.

Moreover, it is supposed to choose each n_h segment, $h = 1, 2, \dots, L$, out of N_h segments of the h^{th} stratum randomly according to the simple random sampling plan. A list of all N_h segments in each stratum h has to be created first, then a random procedure will help to choose n_h of N_h ; N_h is known in advance and n_h is predetermined. If, for some reason, the randomness of the selection of sample segments cannot be preserved, the stratified simple random sample design may lead to some tragic results.

Furthermore, the implementation cost of the stratified simple random sample design can be reduced by using the "rotation sample design" which is introduced in this paper.

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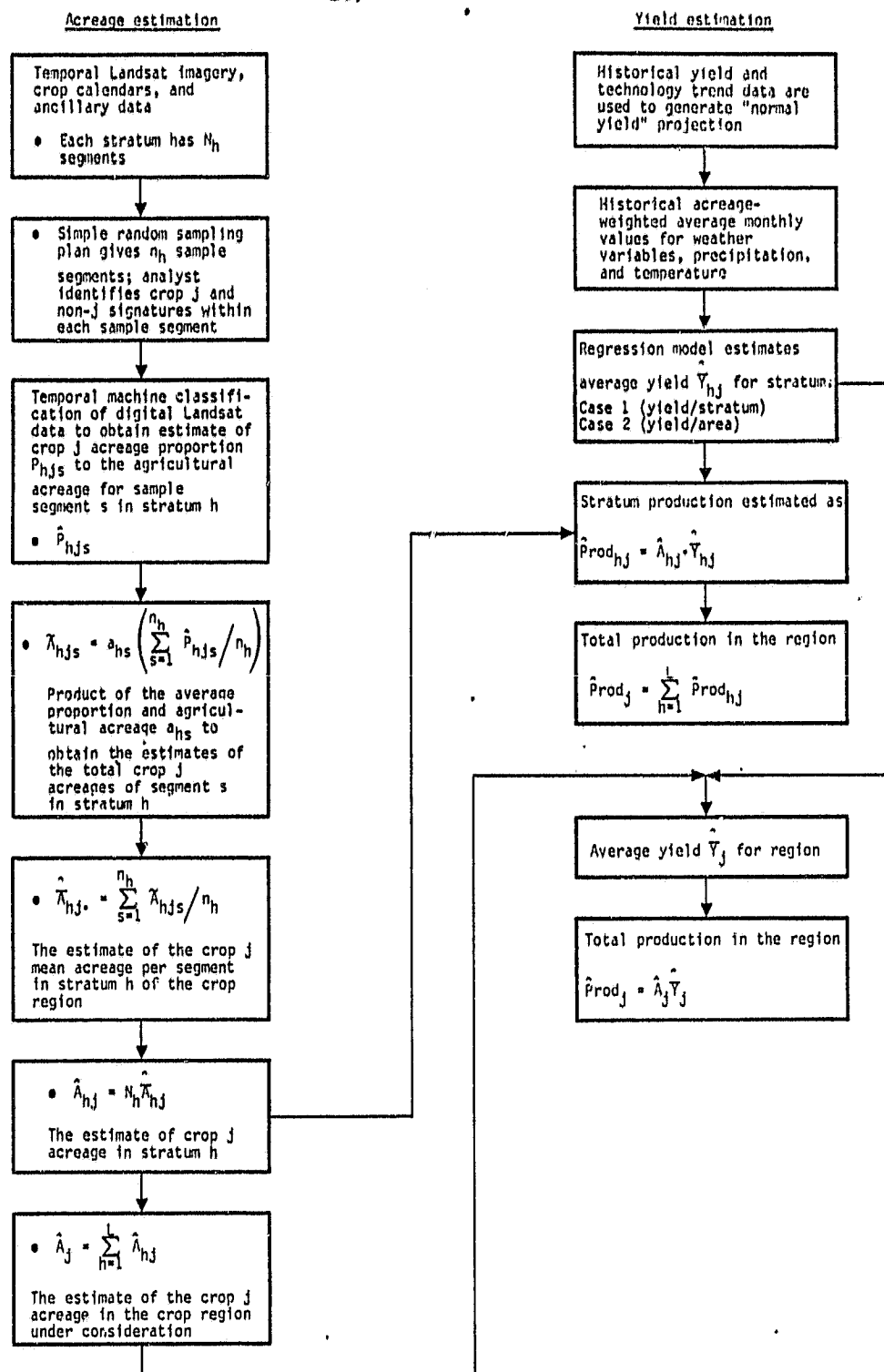


Figure 1-1.- Flow chart of a general estimation approach.

1.2 THE ROTATION SAMPLE DESIGN

The rotation sample design which is presented in this paper was developed to

- (1) improve the accuracy of the crop acreage estimation procedures and
- (2) reduce the cost of the implementation.

The objects of this design are not restricted to any particular crops but, for the sake of practical illustration and testing, wheat has been selected as the crop on which attention will be focused in this paper.

1.2.1 OBJECTIVE OF THE ROTATION SAMPLE DESIGN

The objective of the development of the rotation sample designs for the estimation of wheat acreages is to provide multiyear estimates of wheat acreages which are more accurate than stratified simple random sample estimators. The underlying assumption is that the variation of the wheat acreages of a particular sample unit (named parcel or segment) from year to year is usually less than the variation of wheat acreages between segments within a particular year.

1.2.2 GENERAL APPROACH

The wheat acreage estimation in the design uses the available estimates of "matched" segments, i.e., segments for which estimates are available for two or more years.

Instead of following strictly the simple random sampling plan to obtain n_h out of N_h segments of stratum h , $h = 1, 2, \dots, L$, for each year of estimation, the rotation design will retain each year a fraction of the sample segments and replace the remaining sample segments with new segments in accordance with a rotation pattern of those defined in section 3.2.2.

The estimation theory is based on an additive-mixed analysis-of-variance model with years as fixed effects and parcels as a variable factor.

1.2.3 OBJECTIVE OF THE ROTATION SAMPLE DESIGN TASK

The objective of this effort is to develop a rotation sample design, survey many rotation patterns, and highlight those rotation patterns for which the wheat acreage estimates for the current year have a minimum variance-reduction ratio.

2. STATUS

Because of the specialized nature of this development item, the literature is very limited. Even if the term 'rotation design' is familiar to many statisticians in the field of sample surveys, especially to those who are familiar with the Current Population Survey (CPS) of the Bureau of the Census (ref. 3), the methodology of this particular rotation design is different from that of the CPS rotation design. For instance, the objectives for the AgRISTARS/FCPF project are considerably different from those of the CPS.

2.1 THE BUREAU OF THE CENSUS CPS

2.1.1 DESCRIPTION

Of concern in the CPS is the estimation of social characteristics of the U.S. population and the provision of monthly estimates for these characteristics (i.e., employment, unemployment, income distribution, family characteristics, marital status, migration, education, etc.). The estimates are made from a sample of segments of households which are arranged in "rotation groups" according to the CPS rotation design. There are eight rotation groups (eight systematic subsamples) of segments in each sample. A given rotation group is interviewed over an 8-month period which is divided into two 4-month periods. The segments in the selected rotation group are interviewed during the four consecutive months of the first year, then omitted from the survey during the following eight months, then interviewed again for the same four calendar months of the next year, and, finally, dropped from the survey.

2.1.2 OBJECTIVES

The basic reason for rotating part of the sample each month is to invalidate the problem of uncooperation of respondents.

The other objective in using the CPS rotation design is to provide estimates which take advantage of accumulated information from earlier samples as well as the information from the current sample, which results in smaller variances of estimates.

2.1.3 PROCEDURE

The estimator of a characteristic, for example y , the so-called composite estimator, is a weighted average of the following two estimator components.

- a. The first component is the regular ratio estimate, for example y'_t , based on the entire sample for the month under current consideration time, t .
- b. The second component is the addition of the composite estimate for the preceding month $t - 1$, for example y''_{t-1} , and the estimate of the change in each item from the preceding month to the current month. The estimates used to compute the change are the regular ratio estimates $y'_{t,t-1}$ and $y'_{t-1,t}$ (i.e., the segments that are in the sample in both month $t - 1$ and month t). The second estimate is then

$$y''_{t-1} + y'_{t,t-1} - y'_{t-1,t} \quad (8)$$

Then, the two estimates are combined as a weighted average, with weights summing to one.

In summary, the CPS rotation design composite estimator is

$$y''_t = W(y''_{t-1} + y'_{t,t-1} - y'_{t-1,t}) + (1 - W)y'_t \quad (9)$$

where

y''_t = the composite estimator for month t ,

y'_t = the regular ratio estimator based on the earlier sample for month t ,

$y'_{t,t-1}$ = the regular ratio estimate for month t but is made only from the matched segments which are in month t ,

$y'_{t-1,t}$ = the regular ratio estimate for month $t - 1$, but is made only from the matched segments which are in month $t - 1$.

When there is a positive correlation between y values in two consecutive months, the variances of the changes $(y'_{t,t-1} - y'_{t-1,t})$ will be small, since

$$\begin{aligned} V(y'_{t,t-1} - y'_{t-1,t}) &= V(y'_{t,t-1}) + V(y'_{t-1,t}) - 2 \text{cov}(y'_{t,t-1}, y'_{t-1,t}) \\ &< V(y'_{t,t-1}) + V(y'_{t-1,t}) \end{aligned} \quad (10)$$

Hence, this rotation design with its composite estimator does utilize effectively the information from the earlier and current samples and results in smaller variances due to consideration of the estimates of change.

2.2 A ROTATION SAMPLE DESIGN FOR THE AgRISTARS FCPF PROJECT

The effectiveness of the composite estimator in a rotation design depends upon the year-to-year correlation between crop acreages. Indeed, there is usually a strong positive correlation between the wheat acreages of segments observed in consecutive years, and this similarity seems to imply that the experience gathered in the CPS possibly can be transferred for use in the AgRISTARS FCPF project. Such a transfer, however, is not possible because of the differences in the objectives, the differences in segment construction (the "segment" in the CPS is a geographic segment comprising a number of households), and the substantial differences listed as follows.

- a. The AgRISTARS FCPF time series is yearly and extremely short (two or three years). The CPS time series is monthly.
- b. The FCPF rotation sample design to be developed will have to be suitable to the peculiar trait whereby a considerable number of matched segments (i.e., segments in the sample for two consecutive years) will be lost through cloud cover or other reasons. So, the FCPF rotation design, in this case, has to provide estimators which are capable of dealing with unbalanced segment patterns over a moderate number of years.

Henceforce, the CPS composite estimator is not suitable for use in the AgRISTARS FCPF project. The mixed analysis of variance (ANOVA) models and the associated estimators provided by proper rotation designs were selected and combined to create a new sample design which is custom-made for the AgRISTARS FCPF project.

This sample design will deal suitably with the completely unbalanced matching patterns and will utilize the fact that the variation of the wheat acreages of a particular segment from year to year is usually less than the variation of the wheat acreages of different segments within a particular year.

3. PROCEDURE

Before the main features of the procedure are introduced, it is worthwhile to review some basic background information.

3.1 TARGET POPULATION AND SAMPLING FRAME

3.1.1 POPULATION AND ITS OBJECTS

Agricultural acreages over many selected regions within the United States, Canada, India, U.S.S.R., China, Australia, Argentina, and Brazil is the target population. Wheat is the crop of interest in this paper. However, any other crop such as barley, corn, or soybeans can take the place of wheat provided the crop characteristics satisfy the requirements of the procedure for wheat. One of the requirements is that the variation of the wheat acreages for a particular segment from year to year is usually less than the variation of the wheat acreages of different segments within a particular year.

3.1.2 SAMPLING UNITS AND SAMPLING FRAME

Elementary units are 'pixels' of the earth's surface. Each pixel is a picture element of the imagery provided by the multispectral scanner (MSS) used by Landsat. It corresponds to an area of about 1 acre on the earth's surface.

Sampling units are areal parcels which are clusters of pixels. The sampling unit used for the current AgRISTARS FCPF project is made up of 6- by 3.5-mile segments, the imagery of which comprises approximately 22 932 pixels.

The sampling unit size may be smaller than the 6- by 3.5-mile area as long as it supports the requirement of the procedure mentioned in section 3.1.1 and helps to reduce errors caused by classification, labeling, and calculation of the proportion of wheat within each segment. To explain more fully, classification, in automated analysis of remotely sensed data, is the process of assigning data points to feature classifications by a testing process in which an automated electronic system scans a total image (data set) pixel by pixel and determines whether the spectral properties of each pixel correlate with

those of the subject being classified. Labeling is the process of determining characters which identify an item of data, an area of memory, a record, or a file.

Each crop (wheat) region is stratified into a number of subregions, called strata, which are homogeneous in terms of the crop (wheat) density. This means crop (wheat) densities vary little within a stratum but may vary considerably from stratum to stratum. There is a known number of strata, for example H , within a crop (wheat) region under consideration. Each stratum is comprised of a known number, for example N_h for stratum h , $h = 1, 2, \dots, H$, of sampling units, e.g., 6- by 3.5-mile segments.

The sampling frame is constructed by first covering a crop-growing region of a country by a large grid of segments that are either 6-by-3.5 miles or smaller, then excluding those segments which appeared to show less than 5 percent agriculture. The actual sampling segments (sampling units) are chosen from this sampling frame.

3.2 ROTATION SAMPLE DESIGN

The sample design presented in this paper is named the rotation sample design.

3.2.1 SAMPLING PLAN

After a crop (wheat) region is stratified into L strata and suppose stratum h is comprised of N_h segments, a well-defined allocation procedure will determine the n_h value, number of segments in stratum h to be in the sample. The n_h number will be the same for each year of study, but the segments will be chosen by a specified rotation pattern.

3.2.2 ALLOCATIONS BY ROTATION PATTERNS

With the present one-year-only design, the number of segments n_h allocated to stratum h varies from 1 to a sizable number such as 30 depending on the country of interest. (The number n_h is a number selected from N_h segments through a simple random sampling plan.) In the United States, the number is small; it is one, two, or three when counties are treated as strata.

The rotation sampling plan of this research is confined in the situation in which $n_h = 1, 2$, or 3 segments to be in the sample under consideration in each year. In the situation named Group I, the stratified simple random sample design is taken as the current sample design of the AgRISTARS FCPF project.

The allocation will be in a rotation pattern, as shown in figures 3-1 to 3-7.

In figures 3-1 through 3-4, the notation S is the number n_h of segments per stratum in each year and the notation r is the number of years elapsing before a segment returns to the sample. Thus, figure 3-1 presents $S = 2$ segments per stratum, and each segment returns to the sample after $r = 2$ years.

In figures 3-5, 3-6, and 3-7, the notation $r = \infty$ is used to indicate that a segment will never return to the sample. The expression inside the $[]$ shows the type of pattern, for example $[2 \text{ retained } 1]$ means that of the $S = 2$ segments, one is retained in the next year.

3.2.3 SPECIAL CASES

One special case occurs when some strata have only one segment included in the sample. In this case, the following principles of collapsing can be applied to match one of the specified patterns, i.e., one of those shown in figures 3-1 through 3-7.

One stratum having only one segment included in the sample is collapsed with the other stratum having one of the following:

- a. one sample segment
- b. two sample segments
- c. three sample segments

The collapsibility will be based on the relation of the stratum itself to the other strata which are alike.

		Year number		
Segment number	s \ t	1	2	3
	1	X		X
	2	X	X	
	3		X	X

Figure 3-1.- (S = 2, r = 2) Patterns.

		Year number			
Segment number	s \ t	1	2	3	4
	1	X		X	X
	2	X	X		X
	3	X	X	X	
	4		X	X	X

Figure 3-2.- (S = 3, r = 2) Patterns.

		Year number			
Segment number	s \ t	1	2	3	4
	1	X			X
	2	X	X		
	3		X	X	
	4			X	X

Figure 3-3.- (S = 2, r = 3) Patterns.

		Year number				
Segment number	s \ t	1	2	3	4	5
	1	X			X	X
	2	X	X			X
	3	X	X	X		
	4		X	X	X	
	5			X	X	X

Figure 3-4.- ($S = 3$, $r = 3$) Patterns.

		Year number			
Segment number	s \ t	1	2	3	4
	1	X			
	2	X	X		
	3		X	X	
	4			X	X
	5				X

Figure 3-5.- ($S = 2$, $r = \infty$) Patterns [2 retained 1].

		Year number			
Segment number	$s \backslash t$	1	2	3	4
	1	X			
	2	X	X		
	3	X	X	X	
	4		X	X	X
	5			X	X
	6				X

Figure 3-6.- ($S = 3$, $r = \infty$) Patterns [3 retained 2].

		Year number			
Segment number	$s \backslash t$	1	2	3	4
	1	X			
	2	X	X		
	3	X	X	X	
	4	X	X	X	X
	5		X	X	X
	6			X	X
	7				X

Figure 3-7.- ($S = 4$, $r = \infty$) Patterns [4 retained 3].

3.3 SPECIFIC APPROACH

The mixed analysis of variance model is assumed in this rotation sample design in order to obtain estimators from the completely unbalanced matching patterns. The patterns are likely to arise through losses of segments (due to cloud cover or other reasons), and they differ considerably from any balanced rotation designs.

This paper presents only the basic, simple model which will apply to many different rotation patterns. For each pattern, the variance of the stratum crop acreage estimate for the current year is derived, and an optimal design, which gives a minimum variance of the estimate, is obtained empirically.

3.4 RESOLUTION

3.4.1 MIXED ANALYSIS OF VARIANCE MODEL

3.4.1.1 Basic Model

The estimate of the average wheat acreage per segment is obtained from an infinite analysis of variance model for A_{ts} , the "observed" wheat acreage of segment s in stratum h for year t . The model is written as

$$A_{ts} = \alpha_t + b_s + e_{ts} \quad (11)$$

for $t = 1, 2, \dots, T$ years and $s = 1, 2, \dots, s_h$ segments

where

α_t = average true wheat acreage per segment in year t ; α_t values are fixed year constants,

b_s = true segment variables applicable to all years with the assumption $b \sim N(0, \sigma_b^2)$

e_{ts} = composite segment error variable of segment s in year t with the assumption $e_{ts} \sim N(0, \sigma_e^2)$. This error variable contains two components which are the deviations of the true wheat acreage of segment s of stratum h in year t from the additive formula $\alpha_t + b_s$ and the classification error in A_{ts} .

3.4.1.2 Estimation

3.4.1.2.1 Approach of Estimation for Average Wheat Acreage Per Segment

The development of the basic model (11) to obtain the estimates $\hat{\alpha}_t$ and its variances is based on the assumption of a known constant $\gamma = \sigma_b^2 / \sigma_e^2$ which can be estimated from pilot data. Therefore, the Aitken weighted least-squares estimation will be applicable and utilized to obtain the estimators $\hat{\alpha}_t$ and its variances $\text{Var}(\hat{\alpha}_t)$.

The model shown in equation (11) can be written in the following matrix form:

$$\underline{a} = X\underline{\alpha} + U\underline{b} + I\underline{e} \quad (12)$$

where X and U are the design matrices of equation (11) and are to be defined according to each rotation design in the coming sections, and \underline{a} is the vector of the observed wheat acreages of segments s in years t . In the model, the following values are held.

$$\underline{a} = (A_{11}, A_{21}, \dots, A_{t1}, A_{12}, \dots, A_{t2}, \dots, A_{ts}, \dots, A_{Tsh})'$$

$$\underline{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_t, \dots, \alpha_T)'$$

$$\underline{b} = (b_1, b_2, \dots, b_s, \dots, b_{sh})' ; \underline{b} \sim N_{sh}(\underline{0}, I\sigma_b^2)$$

I = the identity matrix

$$\underline{e} = (e_{11}, e_{21}, \dots, e_{ts}, e_{12}, \dots, e_{t2}, \dots, e_{ts}, \dots, e_{Tsh})' ; \underline{e} \sim N_{Tsh}(\underline{0}, I\sigma_e^2)$$

Since $(\underline{\alpha})$ is a vector of fixed-year constants, equation (12) gives

$$\begin{aligned} \text{Var}(\underline{a}) &= \text{Var}(U\underline{b} + I\underline{e}) \\ &= U \text{Var}(\underline{b})U' + \text{Var}(\underline{e}) \\ &= UU' \sigma_b^2 + I\sigma_e^2 \\ &= \left(I + \frac{\sigma_b^2}{\sigma_e^2} UU' \right) \sigma_e^2 \end{aligned} \quad (13)$$

or,

$$\text{Var}(\underline{a}) \equiv H\sigma_e^2 \quad (14)$$

where

$$\gamma \equiv \sigma_b^2 / \sigma_e^2 \quad (15)$$

$$H \equiv I + \gamma UU' \quad (16)$$

with γ being regarded as a known constant. H is supposed to be invertible.

Being multiplied both sides by $H^{-1/2}$, equation (12) gives

$$H^{-1/2}\underline{a} = H^{-1/2}\underline{X}\underline{\alpha} + H^{-1/2}U\underline{b} + H^{-1/2}\underline{e} \quad (17)$$

Put

$$H^{-1/2}\underline{a} \equiv \underline{a}^*$$

$$H^{-1/2}\underline{X} \equiv \underline{X}^*$$

$$H^{-1/2}U \equiv U^*$$

$$H^{-1/2}\underline{e} \equiv \underline{e}^*$$

then equation (17) can be written as

$$\underline{a}^* = \underline{X}^*\underline{\alpha} + U^*\underline{b} + \underline{e}^* \quad (18)$$

where

$$\begin{aligned} \text{Var}(\underline{a}^*) &= \text{Var}(H^{-1/2}\underline{a}) \\ &= H^{-1/2} \text{Var}(\underline{a}) H^{-1/2} \\ &= H^{-1/2} H H^{-1/2} \sigma_e^2 \\ &= I \sigma_e^2 \end{aligned} \quad (19)$$

and

$$\begin{aligned}
 \text{Var}(U^*b + e^*) &= \text{Var}\left(H^{-1/2}Ub + H^{-1/2}e\right) \\
 &= H^{-1/2}U \text{Var}(b)U'H^{-1/2} + H^{-1/2} \text{Var}(e)H^{-1/2} \\
 &= H^{-1/2}\left(H\sigma_e^2 - I\sigma_e^2\right)H^{-1/2} + H^{-1/2}\sigma_e^2H^{-1/2}
 \end{aligned} \tag{20}$$

because equations (13) and (14) imply

$$\begin{aligned}
 U \text{Var}(b)U' &= UU'\sigma_b^2 \\
 &= H\sigma_e^2 - I\sigma_e^2
 \end{aligned} \tag{21}$$

Hence, equations (20) and (21) imply

$$\begin{aligned}
 \text{Var}(U^*b + e^*) &= H^{-1/2}(H - I)H^{-1/2}\sigma_e^2 + H^{-1/2}\sigma_e^2H^{-1/2} \\
 &= H^{-1/2}(H - I + I)H^{-1/2}\sigma_e^2 \\
 &= H^{-1/2}HH^{-1/2}\sigma_e^2 \\
 &= I\sigma_e^2
 \end{aligned} \tag{22}$$

Equations (19) and (22) give the following conclusion.

$$\text{Var}(\tilde{a}^*) = \text{Var}(U^*b + e^*) \tag{23}$$

So, the model shown in equation (18),

$$\tilde{a}^* = X^*\alpha + U^*b + e^* \tag{24}$$

has the following property

$$\text{Var}(\tilde{a}^*) = \text{Var}(U^*b + e^*) = I\sigma_e^2 \tag{25}$$

It is well-known linear model with known formulae which are deduced as follows. The normal equation of model (24) is

$$X^{*'} \underline{a}^* = (X^{*'} X^*) \hat{\underline{g}} \quad (26)$$

That means

$$(H^{-1/2} X)' H^{-1/2} \underline{a} = (H^{-1/2} X)' (H^{-1/2} X) \hat{\underline{g}}$$

i.e.,

$$X' H^{-1} \underline{a} = X' H^{-1} X \hat{\underline{g}} \quad (27)$$

The solution is

$$\hat{\underline{g}} = (X' H^{-1} X)^{-1} X' H^{-1} \underline{a} \quad (28)$$

which is the best linear unbiased estimator (BLUE) of \underline{g} by the Gauss-Markoff theorem and is also the maximum likelihood estimator. From pilot data, γ can be estimated as $\hat{\gamma} = \hat{\sigma}_b^2 / \hat{\sigma}_e^2$ which is substituted in equations (27) and (28). If a consistent estimator of γ is employed, it can be shown (see ref. 4) that equation (28) is still consistent and the variance of the estimator, which will be given below in equation (29), still applies asymptotically with $\hat{\gamma} \rightarrow \gamma$ as an error of order for magnitude $\text{Var}(\hat{\underline{g}}) \text{Var}(\hat{\gamma})$. Equation (28) gives

$$\begin{aligned} \text{Var}(\hat{\underline{g}}) &= \text{Var}[(X' H^{-1} X)^{-1} X' H^{-1} \underline{a}] \\ &= (X' H^{-1} X)^{-1} X' H^{-1} \text{Var}(\underline{a}) H^{-1} X (X' H^{-1} X)^{-1} \\ &= (X' H^{-1} X)^{-1} (X' H^{-1} H H^{-1} X) (X' H^{-1} X)^{-1} \sigma_e^2 \\ &= (X' H^{-1} X)^{-1} \sigma_e^2 \end{aligned}$$

that is,

$$\text{Var}(\hat{\underline{g}}) = (X' H^{-1} X)^{-1} \sigma_e^2 \quad (29)$$

3.4.1.2.2 Approach of Estimation for Stratum Wheat Acreage

From equations (28) and (29), the estimate of average wheat acreage per segment within stratum h at the current year indexed $t = T$ and its variance $\hat{\alpha}_{Th}$ and $\text{Var}(\hat{\alpha}_{Th})$ are obtained.

Then the estimate of stratum h wheat acreage at the year T will be

$$\hat{A}_{Th} = N_h \hat{\alpha}_{Th} \quad (30)$$

with its variance formula

$$\text{Var}(\hat{A}_{Th}) = N_h^2 \text{Var}(\hat{\alpha}_{Th}) \quad (31)$$

where $\text{Var}(\hat{\alpha}_{Th})$ is deduced from equation (29).

3.4.1.2.3 Flow Chart of a General Approach of Estimation

As a general summary, the flow chart shown as figure 3-8 presents a general approach of crop acreage estimation and production.

3.4.2 RESOLUTION FOR ($S = 2, r = 2$) ROTATION PATTERNS

The rotation pattern ($S = 2, r = 2$) is presented in figure 3-1, which is repeated as follows.

		Year number		
Segment number	s \ t	1	2	3
	1	X		X
	2	X	X	
	3		X	X

($S = 2, r = 2$) Patterns

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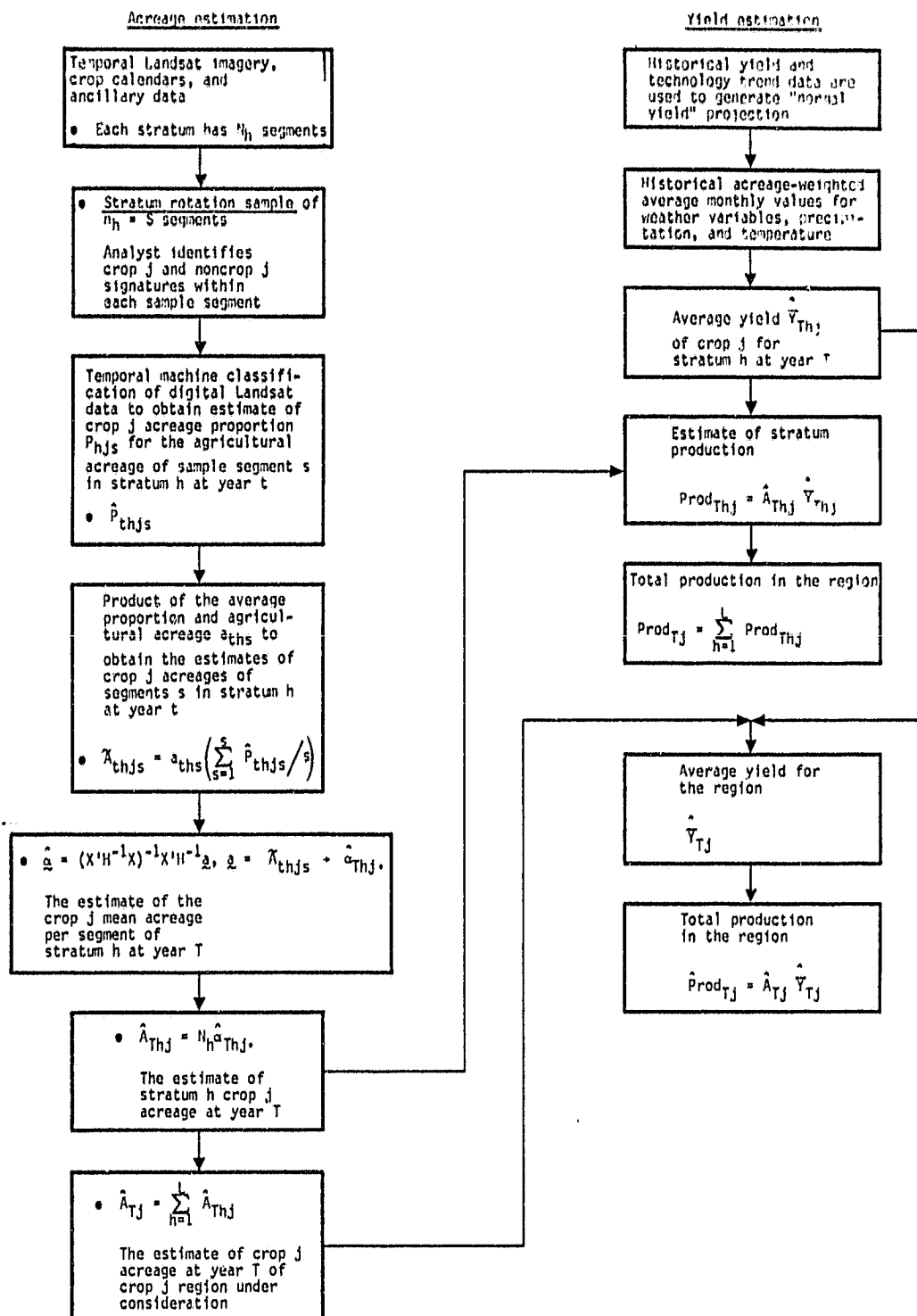


Figure 3-8.- Flow chart of a general estimation approach.

The design matrices of the related model, which is shown in equation (12), are

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (32)$$

and

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad (33)$$

Hence

$$H \equiv I + \gamma U U' = \begin{bmatrix} 1 + \gamma & \gamma & 0 & 0 & 0 & 0 \\ \gamma & 1 + \gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 + \gamma & \gamma & 0 & 0 \\ 0 & 0 & \gamma & 1 + \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 + \gamma & \gamma \\ 0 & 0 & 0 & 0 & \gamma & 1 + \gamma \end{bmatrix} \quad (34)$$

or, with $a \equiv \gamma/(1 + \gamma)$,

$$H = (1 + \gamma) \begin{bmatrix} 1 & a & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & a & 0 & 0 \\ 0 & 0 & a & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & a \\ 0 & 0 & 0 & 0 & a & 1 \end{bmatrix} \quad (35)$$

Then,

$$H^{-1} = \frac{1}{(1 + \gamma)(1 - a^2)} \begin{bmatrix} 1 & -a & 0 & 0 & 0 & 0 \\ -a & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -a & 0 & 0 \\ 0 & 0 & -a & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 0 & -a & 1 \end{bmatrix} \quad (36)$$

$$= \frac{1}{1 + 2\gamma} \begin{bmatrix} 1 + \gamma & -\gamma & 0 & 0 & 0 & 0 \\ -\gamma & 1 + \gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 + \gamma & -\gamma & 0 & 0 \\ 0 & 0 & -\gamma & 1 + \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 + \gamma & -\gamma \\ 0 & 0 & 0 & 0 & -\gamma & 1 + \gamma \end{bmatrix}$$

$$X'H^{-1}X = \frac{1}{1 + 2\gamma} \begin{bmatrix} 2(1 + \gamma) & -\gamma & -\gamma \\ -\gamma & 2(1 + \gamma) & -\gamma \\ -\gamma & -\gamma & 2(1 + \gamma) \end{bmatrix} \quad (37)$$

and

$$(X'H^{-1}X)^{-1} = (1 + 2\gamma) \begin{bmatrix} 2(1 + \gamma) & -\gamma & -\gamma \\ -\gamma & 2(1 + \gamma) & -\gamma \\ -\gamma & -\gamma & 2(1 + \gamma) \end{bmatrix}^{-1} \quad (38)$$

The inverse matrix in the second member of the above equation is found by applying the result given in appendix B where

$$(a_{ij}) = \begin{bmatrix} a & b & \dots & b \\ b & a & & \\ \vdots & & \ddots & \vdots \\ b & \dots & & a \end{bmatrix}$$

where, in this case,

$$a = 2(1 + \gamma)$$

$$b = -\gamma$$

$$n = 3$$

and

$$(a_{ij})^{-1} = (a^{ij}) \quad (39)$$

hence,

$$a^{ii} = \frac{a + (n-2)b}{[a + (n-1)b](a-b)} \quad (40)$$

that is,

$$\begin{aligned} a^{ii} &= \frac{2(1+\gamma) + (-\gamma)}{[2(1+\gamma) + 2(-\gamma)][2(1+\gamma) + \gamma]} \\ &= \frac{2+\gamma}{2(2+3\gamma)} \end{aligned} \quad (41)$$

and

$$a^{ij} = \frac{-b}{[a + (n-1)b](a-b)} \quad (42)$$

$$a^{ij} = \frac{\gamma}{2(2+3\gamma)} \quad (43)$$

So,

$$(X'H^{-1}X)^{-1} = \frac{1+2\gamma}{2(2+3\gamma)} \begin{bmatrix} 2+\gamma & \gamma & \gamma \\ \gamma & 2+\gamma & \gamma \\ \gamma & \gamma & 2+\gamma \end{bmatrix} \quad (44)$$

For this rotation pattern, the basic model

$$\underline{a} = X\underline{\alpha} + U\underline{b} + I\underline{e}$$

can be written as

$$A_{ts} = \alpha_t + b_s + e_{ts} \quad (45)$$

where

$$t = 1, 2, 3$$

$$s = 1, 2, 3$$

$$\underline{\alpha} = (\alpha_1, \alpha_2, \alpha_3)'$$

$$\underline{b} = (b_1, b_2, b_3)'$$

That is,

$$\begin{bmatrix} A_{11} \\ A_{31} \\ A_{12} \\ A_{22} \\ A_{23} \\ A_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + \begin{bmatrix} e_{11} \\ e_{31} \\ e_{12} \\ e_{22} \\ e_{23} \\ e_{33} \end{bmatrix} \quad (46)$$

where

$$\underline{a} = (A_{11}, A_{31}, A_{12}, A_{22}, A_{23}, A_{33})' \quad (47)$$

By step 4 of the acreage estimation in figure 3-8, the estimates of the A_{ts} 's are known as \tilde{A}_{ts} ; hence, the estimates of the α_t 's are deduced from equation (28) which is now

$$\hat{\underline{\alpha}} = (X'H^{-1}X)^{-1}X'H^{-1}\underline{a}^0 \quad (48)$$

where

$$\underline{a}^0 = (\tilde{A}_{11}, \tilde{A}_{31}, \tilde{A}_{12}, \tilde{A}_{22}, \tilde{A}_{23}, \tilde{A}_{33})' \quad (49)$$

and where

$$H = I + \hat{\gamma}UU', \text{ with } \hat{\gamma} = \hat{\sigma}_b^2 / \hat{\sigma}_e^2 \text{ estimated from the pilot data,}$$

U is given by equation (33),

$(X'H^{-1}X)^{-1}$ is given by equation (44),

X is given by equation (32), and

H^{-1} is given by equation (34) with $\hat{\gamma}$.

The variances of the estimates are deduced from equation (29) which is now

$$\text{Var}(\hat{\alpha}) = (X'H^{-1}X)^{-1}\sigma_e^2 = \sigma_e^2 \frac{1 + 2\hat{\gamma}}{2(2 + 3\hat{\gamma})} \begin{bmatrix} 2 + \hat{\gamma} & \hat{\gamma} & \hat{\gamma} \\ \hat{\gamma} & 2 + \hat{\gamma} & \hat{\gamma} \\ \hat{\gamma} & \hat{\gamma} & 2 + \hat{\gamma} \end{bmatrix} \quad (50)$$

where $\hat{\gamma} = \hat{\sigma}_b^2 / \hat{\sigma}_e^2$, as estimated from the pilot data.

This means

$$\text{Var } \hat{\alpha}_t = \frac{(1 + 2\hat{\gamma})(2 + \hat{\gamma})}{2(2 + 3\hat{\gamma})} \sigma_e^2 \quad (51)$$

for $t = 1, 2, 3$, with $T = 3$.

Emphatically, at the last year indexed $t = T$,

$$\boxed{\text{Var}(\hat{\alpha}_t) = \frac{(1 + 2\hat{\gamma})(2 + \hat{\gamma})}{2(2 + 3\hat{\gamma})} \sigma_e^2} \quad (52)$$

With respect to this, note the following:

- a. The datum T represents the total number of years utilized for the multiyear estimator $\hat{\alpha}_T$. In the case of rotation with "return" (i.e., $r < \infty$), the maximum value of T is $r + 1$ since this corresponds to a complete cycle of the rotation.
- b. Remarkably, in the case of $T = 3$, if the rotation pattern cannot be as shown in figure 3-1, but instead will be shown in figure 3-9.

		Year number		
Segment number	s \ t	1	2	3
	1	X	X	
	2		X	X
	3	X		X

Figure 3-9.- (S = 2, r = 2) Patterns, showing inverse of pattern given figure 3-1.

Then, the design matrices X and U will be as follows.

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad (53)$$

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad (54)$$

therefore,

$$H = \begin{bmatrix} 1 + \gamma & \gamma & 0 & 0 & 0 & 0 \\ \gamma & 1 + \gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 + \gamma & \gamma & 0 & 0 \\ 0 & 0 & \gamma & 1 + \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 + \gamma & \gamma \\ 0 & 0 & 0 & 0 & \gamma & 1 + \gamma \end{bmatrix} \quad (55)$$

$$H^{-1} = \frac{1}{1+2\gamma} \begin{bmatrix} 1+\gamma & -\gamma & 0 & 0 & 0 & 0 \\ -\gamma & 1+\gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+\gamma & -\gamma & 0 & 0 \\ 0 & 0 & -\gamma & 1+\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\gamma & -\gamma \\ 0 & 0 & 0 & 0 & -\gamma & 1+\gamma \end{bmatrix} \quad (56)$$

$$X'H^{-1}X = \frac{1}{1+2\gamma} \begin{bmatrix} 2(1+\gamma) & -\gamma & -\gamma \\ -\gamma & 2(1+\gamma) & -\gamma \\ -\gamma & -\gamma & 2(1+\gamma) \end{bmatrix} \quad (57)$$

which is the same as equation (36). Hence, equation (52) is obtained.

$$\text{Var}(\hat{\alpha}_T) = \frac{(1+2\hat{\gamma})(2+\hat{\gamma})}{2(2+3\hat{\gamma})} \sigma_e^2 \quad (58)$$

However, the components of \underline{a}^0 , which are given in equation (48), will be different in the case of figure 3-9, in which

$$\underline{a}^0 = (\tilde{A}_{11}, \tilde{A}_{21}, \tilde{A}_{22}, \tilde{A}_{32}, \tilde{A}_{33}, \tilde{A}_{13})' \quad (59)$$

Components \tilde{A}_{12} , \tilde{A}_{23} , \tilde{A}_{31} are no longer needed, and \tilde{A}_{13} , \tilde{A}_{21} , and \tilde{A}_{32} are needed instead.

3.4.3 RESOLUTION FOR (S = 3, r = 2) ROTATION PATTERNS

The rotation pattern (S = 3, r = 2) is presented in figure 3-2, which is repeated as follows.

		Year number			
Segment number	s \ t	1	2	3	4
	1	X		X	X
	2	X	X		X
	3	X	X	X	
	4		X	X	X

(S = 3, r = 2) Patterns

The design matrices of the model shown as equation (12) are as follows.

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (60)$$

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (61)$$

Then,

$$H = I + \gamma U U' = \begin{bmatrix} 1+\gamma & \gamma & \gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma & 1+\gamma & \gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma & \gamma & 1+\gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\gamma & \gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma & 1+\gamma & \gamma & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma & \gamma & 1+\gamma & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1+\gamma & \gamma & \gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma & 1+\gamma & \gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma & \gamma & 1+\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+\gamma & \gamma & \gamma \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma & \gamma & 1+\gamma & \gamma \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma & \gamma & \gamma & 1+\gamma \end{bmatrix} \quad (62)$$

and

$$H^{-1} = \frac{1}{1+3\gamma} \begin{bmatrix} 1+2\gamma & -\gamma & -\gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\gamma & 1+2\gamma & -\gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\gamma & -\gamma & 1+2\gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+2\gamma & -\gamma & -\gamma & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\gamma & 1+2\gamma & -\gamma & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\gamma & -\gamma & 1+2\gamma & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1+2\gamma & -\gamma & -\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\gamma & 1+2\gamma & -\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\gamma & -\gamma & 1+2\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+2\gamma & -\gamma & -\gamma \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\gamma & 1+2\gamma & -\gamma \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\gamma & -\gamma & 1+2\gamma \end{bmatrix} \quad (63)$$

since, by the result given by appendix B, the inverses of the 3-by-3 matrices in equation (62) have their components

$$a^{ij} = \frac{(1+\gamma) + (3-2)\gamma}{[(1+\gamma) + (3-1)\gamma][(1+\gamma) - \gamma]} = \frac{1+2\gamma}{1+3\gamma} \quad (64)$$

and

$$a^{ij} = \frac{-\gamma}{1+3\gamma} \quad (65)$$

So,

$$X'H^{-1}X = \frac{1}{1+3\gamma} \begin{bmatrix} 3(1+2\gamma) & -2\gamma & -2\gamma & -2\gamma \\ -2\gamma & 3(1+2\gamma) & -2\gamma & -2\gamma \\ -2\gamma & -2\gamma & 3(1+2\gamma) & -2\gamma \\ -2\gamma & -2\gamma & -2\gamma & 3(1+2\gamma) \end{bmatrix} \quad (66)$$

which is of the form shown in appendix A. In addition, its inverse will be

$$(X'H^{-1}X)^{-1} = \frac{1+3\gamma}{3(3+8\gamma)} \begin{bmatrix} 3+2\gamma & 2\gamma & 2\gamma & 2\gamma \\ 2\gamma & 3+2\gamma & 2\gamma & 2\gamma \\ 2\gamma & 2\gamma & 3+2\gamma & 2\gamma \\ 2\gamma & 2\gamma & 2\gamma & 3+2\gamma \end{bmatrix} \quad (67)$$

Vector \underline{a} in the basic model, as shown in equation (12), for this rotation pattern is

$$\underline{a} = (A_{11}, A_{31}, A_{41}, A_{12}, A_{22}, A_{42}, A_{13}, A_{23}, A_{33}, A_{24}, A_{34}, A_{44})' \quad (68)$$

With the \tilde{A}_{ts} 's being the estimates of the A_{ts} 's of the above components, the $\hat{\alpha}_t$'s of the estimates are deduced from

$$\hat{\underline{a}} = (X'H^{-1}X)^{-1}X'H^{-1}\underline{a}^0 \quad (69)$$

where

$$\underline{a}^0 = (\tilde{A}_{11}, \tilde{A}_{31}, \tilde{A}_{41}, \tilde{A}_{12}, \tilde{A}_{22}, \tilde{A}_{42}, \tilde{A}_{13}, \tilde{A}_{23}, \tilde{A}_{33}, \tilde{A}_{24}, \tilde{A}_{34}, \tilde{A}_{44})' \quad (70)$$

And, the variances of the estimates are deduced from the variance-covariance matrix

$$\begin{aligned} \text{Var}(\hat{\underline{a}}) &= (X'H^{-1}X)^{-1}\sigma_e^2 \\ &= \sigma_e^2 \frac{1 + 3\hat{\gamma}}{3(3 + 8\hat{\gamma})} \begin{bmatrix} 3 + 2\hat{\gamma} & 2\hat{\gamma} & 2\hat{\gamma} & 2\hat{\gamma} \\ 2\hat{\gamma} & 3 + 2\hat{\gamma} & 2\hat{\gamma} & 2\hat{\gamma} \\ 2\hat{\gamma} & 2\hat{\gamma} & 3 + 2\hat{\gamma} & 2\hat{\gamma} \\ 2\hat{\gamma} & 2\hat{\gamma} & 2\hat{\gamma} & 3 + 2\hat{\gamma} \end{bmatrix} \quad (71) \end{aligned}$$

where $\hat{\gamma} = \hat{\sigma}_b^2 / \hat{\sigma}_e^2$, as estimated from the pilot data.

That means

$$\text{Var}(\hat{\alpha}_t) = \frac{(1 + 3\hat{\gamma})(3 + 2\hat{\gamma})}{3(3 + 8\hat{\gamma})} \sigma_e^2 \quad (72)$$

for $t = 1, 2, 3, 4$, and $T = 4$, or

$$\boxed{\text{Var}(\hat{\alpha}_T) = \frac{(1 + 3\hat{\gamma})(3 + 2\hat{\gamma})}{3(3 + 8\hat{\gamma})} \sigma_e^2} \quad (73)$$

3.4.4 RESOLUTION FOR ($S = 2, r = 3$) ROTATION PATTERNS

The rotation pattern ($S = 2, r = 3$) is presented in figure 3-3, which is repeated as follows.

		Year number			
Segment number	s \ t	1	2	3	4
	1	X			X
	2	X	X		
	3		X	X	
	4			X	X

(S = 2, r = 3) Patterns

The design matrices related to the model shown in equation (12) are

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (74)$$

and

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (75)$$

Then, with $a = \gamma/(1 + \gamma)$, and $(1 - a^2)(1 + \gamma) = (1 + 2\gamma)/(1 + \gamma)$,

$$H = (1 + \gamma) \begin{bmatrix} 1 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & a \\ 0 & 0 & 0 & 0 & 0 & 0 & a & 1 \end{bmatrix} \quad (76)$$

$$H^{-1} = \frac{1 + \gamma}{1 + 2\gamma} \begin{bmatrix} 1 & -a & 0 & 0 & 0 & 0 & 0 & 0 \\ -a & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -a & 0 & 0 & 0 & 0 \\ 0 & 0 & -a & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -a & 0 & 0 \\ 0 & 0 & 0 & 0 & -a & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 0 & 0 & 0 & -a & 1 \end{bmatrix} \quad (77)$$

and

$$\begin{aligned} X'H^{-1}X &= \frac{1 + \gamma}{1 + 2\gamma} \begin{bmatrix} 2 & -a & 0 & -a \\ -a & 2 & -a & 0 \\ 0 & -a & 2 & -a \\ -a & 0 & -a & 2 \end{bmatrix} \\ &= \frac{1 + \gamma}{1 + 2\gamma} \text{Circ}(2, -a, 0, -a) \\ &= \frac{1}{1 + 2\gamma} \text{Circ}(2(1 + \gamma), -\gamma, 0, -\gamma) \end{aligned} \quad (78)$$

Appendix C shows the elements of the inverse of the circulant in $X'H^{-1}X$, which for $T = 4$, deduces

$$\text{Var } \hat{\alpha}_T = \frac{1 + 2\gamma}{8} \left(1 + \frac{1}{1 + 2\gamma} - \frac{2}{1 + \gamma} \right) \sigma_e^2 \quad (79)$$

where $\hat{\gamma} = \hat{\sigma}_b^2 / \hat{\sigma}_e^2$, as estimated from the pilot data.

3.4.5 RESOLUTION FOR (S = 3, r = 3) ROTATION PATTERNS

The rotation pattern (S = 3, r = 2) is presented in figure 3-4, which is repeated as follows.

		Year number				
Segment number	s \ t	1	2	3	4	5
	1	X			X	X
	2	X	X			X
	3	X	X	X		
	4		X	X	X	
	5			X	X	X

(S = 3, r = 3) Pattern

The design matrices related to the model shown in equation (12) are as follows.

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (80)$$

and

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (81)$$

Then, with $a = \gamma/(1 + \gamma)$ and $U = (1 - a)(1 + 2a) = (1 + 3\gamma)/(1 + \gamma)^2$

$$H = (1 + \gamma) \begin{bmatrix} 1 & a & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a & 1 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a & a & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & a & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 1 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & a & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & a & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & 1 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & a & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & a & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 1 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & a & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 1 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & a & 1 & 1 \end{bmatrix} \quad (82)$$

and

$$H^{-1} = \frac{1}{(1+\gamma)U} \begin{bmatrix} 1+a & -a & -a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a & 1+a & -a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a & -a & 1+a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+a & -a & -a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 1+a & -a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & -a & 1+a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1+a & -a & -a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -a & 1+a & -a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -a & -a & 1+a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+a & -a & -a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a & 1+a & -a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a & -a & 1+a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+a & -a & -a \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a & 1+a & -a \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a & -a & 1+a \end{bmatrix} \quad (83)$$

Additionally,

$$X'H^{-1}X = \frac{1+\gamma}{1+3\gamma} \begin{bmatrix} 3(1+a) & -2a & -a & -a & -2a \\ -2a & 3(1+a) & -2a & -a & -a \\ -a & -2a & 3(1+a) & -2a & -a \\ -a & -a & -2a & 3(1+a) & -2a \\ -2a & -a & -a & -2a & 3(1+a) \end{bmatrix}$$

$$= \frac{\gamma}{1+3\gamma} \begin{bmatrix} \frac{3}{\gamma}(1+2\gamma) & -2 & -1 & -1 & -2 \\ -2 & \frac{3}{\gamma}(1+2\gamma) & -2 & -1 & -1 \\ -1 & -2 & \frac{3}{\gamma}(1+2\gamma) & -2 & -1 \\ -1 & -1 & -2 & \frac{3}{\gamma}(1+2\gamma) & -2 \\ -2 & -1 & -1 & -2 & \frac{3}{\gamma}(1+2\gamma) \end{bmatrix} \quad (84)$$

that is $X'H^{-1}X = \frac{\gamma}{1+3\gamma} \text{Circ}\left(\frac{3}{\gamma}(1+2\gamma), -2, -1, -1, -2\right)$ (85)

By appendix C,

$$(X'H^{-1}X)_{55}^{-1} = \frac{1}{5} \left(\frac{1}{\gamma} + 3 \right) \sum_{\ell=1}^{T=5} \frac{e^{-i\gamma(\ell-1)2\pi/5}}{\lambda_{\ell}} \quad (86)$$

where

$$\lambda_2 = \sum_{k=1}^{T=5} a(k) e^{-i(1-1)(k-1)2\pi/5} \quad (87)$$

Hence, with

$$\left. \begin{aligned} a(1) &= \frac{3}{\gamma}(1 + 2\gamma) = \frac{3}{\gamma} + 6 \\ a(2) &= -2 = a(5) \\ a(3) &= -1 = a(4) \end{aligned} \right\} \quad (88)$$

then

$$\lambda_1 = \sum_{k=1}^5 a(k) = 3/\gamma \quad (89)$$

$$\begin{aligned} \lambda_2 &= a(1) + a(2)e^{-i2\pi/5} + a(3)e^{-i2 \times 2\pi/5} + a(4)e^{-i3 \times 2\pi/5} + a(5)e^{-i4 \times 2\pi/5} \\ &= \frac{3}{\gamma} + 6 - 2(.309 - .951i) - (-.809 - .588i) - (-.809 + .588i) - 2(.309 + .951i) \\ &= \frac{3}{\gamma} + 6 - 4(.309) + 2(.809) \\ &= \frac{3}{\gamma} + 6.382 \end{aligned} \quad (90)$$

$$\begin{aligned} \lambda_3 &= a(1) + a(2)e^{-i2 \times 2\pi/5} + a(3)e^{-i4 \times 2\pi/5} + a(4)e^{-i6 \times 2\pi/5} + a(5)e^{-i8 \times 2\pi/5} \\ &= \frac{3}{\gamma} + 6 - 2(-.809 - .588i) - (.309 + .951i) - (.309 - .951i) - 2(-.809 + .588i) \\ &= \frac{3}{\gamma} + 6 + 4(.809) - 2(.309) \\ &= \frac{3}{\gamma} + 8.618 \end{aligned} \quad (91)$$

Similarly,

$$\lambda_4 = \frac{3}{\gamma} + 8.618 = \lambda_3 \quad (92)$$

and

$$\lambda_5 = \frac{3}{\gamma} + 6.382 = \lambda_2 \quad (93)$$

Then,

$$\begin{aligned} (X^H H^{-1} X)^{-1}_{55} &= \frac{1}{5} \left(\frac{1}{\gamma} + 3 \right) \left\{ \frac{1}{\lambda_1} + \frac{e^{-i8 \times 2\pi/5}}{\lambda_2} + \frac{e^{-i16 \times 2\pi/5}}{\lambda_3} + \frac{e^{-i24 \times 2\pi/5}}{\lambda_4} + \frac{e^{-i32 \times 2\pi/5}}{\lambda_5} \right\} \\ &= \frac{1}{5} \left(\frac{1}{\gamma} + 3 \right) \left\{ \frac{\gamma}{3} + \frac{-.809 + .588i}{\frac{3}{\gamma} + 6.382} + \frac{.309 - .951i}{\frac{3}{\gamma} + 8.618} + \frac{.309 + .951i}{\frac{3}{\gamma} + 8.618} + \frac{-.809 - .588i}{\frac{3}{\gamma} + 6.382} \right\} \end{aligned}$$

That is,

$$(X'H^{-1}X)_{55}^{-1} = \frac{1}{5} \left(\frac{1}{\gamma} + 3 \right) \left\{ \frac{\gamma}{3} - \frac{1.618}{\frac{3}{\gamma} + 6.382} + \frac{0.618}{\frac{3}{\gamma} + 8.618} \right\} \quad (94)$$

and

$$(X'H^{-1}X)_{55}^{-1} = \frac{1}{5}(1 + 3\gamma) \left\{ \frac{1}{3} + \frac{0.618}{3 + 8.018\gamma} - \frac{1.618}{3 + 6.382\gamma} \right\} \quad (95)$$

Then, for $T = 5$,

$$\text{Var } \hat{\alpha}_T = \frac{1}{5}(1 + 3\hat{\gamma}) \left(\frac{1}{3} + \frac{0.618}{3 + 8.618\hat{\gamma}} - \frac{1.618}{3 + 6.382\hat{\gamma}} \right) \sigma_e^2 \quad (96)$$

where $\hat{\gamma} = \hat{\sigma}_b^2 / \hat{\sigma}_e^2$, as estimated from the pilot data.

3.4.6 RESOLUTION FOR ($S = 2$, $r = \infty$) ROTATION PATTERNS

3.4.6.1 Case of $T = 2$

The rotation pattern ($S = 2$, $r = \infty$; $T = 2$) is presented below in figure 3-10.

		Year number	
Segment number	s \ t	1	2
	1	X	
	2	X	X
	3		X

Figure 3-10.- ($S = 2$, $r = \infty$) Patterns [2 retained 1; $T = 2$].

The design matrices related to the model shown in equation (12) are

$$X = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \quad (97)$$

and

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (98)$$

Then,

$$H = I + \gamma U U' = \begin{bmatrix} 1 + \gamma & 0 & 0 & 0 \\ 0 & 1 + \gamma & \gamma & 0 \\ 0 & \gamma & 1 + \gamma & 0 \\ 0 & 0 & 0 & 1 + \gamma \end{bmatrix} \quad (99)$$

and

$$H^{-1} = \frac{1}{1 + 2\gamma} \begin{bmatrix} 1 + 2\gamma & 0 & 0 & 0 \\ 0 & (1 + \gamma)^2 & -\gamma(1 + \gamma) & 0 \\ 0 & -\gamma(1 + \gamma) & (1 + \gamma)^2 & 0 \\ 0 & 0 & 0 & 1 + 2\gamma \end{bmatrix} \quad (100)$$

and

$$X' H^{-1} X = \frac{1}{(1 + \gamma)(1 + 2\gamma)} \begin{bmatrix} (1 + 2\gamma) + (1 + \gamma)^2 & -\gamma(1 + \gamma) \\ -\gamma(1 + \gamma) & (1 + 2\gamma) + (1 + \gamma)^2 \end{bmatrix} \quad (101)$$

So,

$$(X' H^{-1} X)^{-1} = \frac{(1 + \gamma)(1 + 2\gamma)}{[(1 + 2\gamma) + (1 + \gamma)^2]^2 - \gamma^2(1 - \gamma)^2} \begin{bmatrix} (1 + 2\gamma) + (1 + \gamma)^2 & \gamma(1 + \gamma) \\ \gamma(1 + \gamma) & (1 + 2\gamma) + (1 + \gamma)^2 \end{bmatrix} \quad (102)$$

Vector \underline{a} in the basic model shown in equation (12) for this rotation pattern is

$$\underline{a} = (A_{11}, A_{12}, A_{22}, A_{23})' \quad (103)$$

With the \tilde{A}_{ts} 's being the estimates of the A_{ts} 's of the above components the $\hat{\alpha}_t$'s of the estimates are deduced from

$$\hat{\underline{\alpha}} = (X' H^{-1} X)^{-1} X' H^{-1} \underline{a}^0 \quad (104)$$

where,

$$\underline{\hat{a}}^0 = (\tilde{A}_{11}, \tilde{A}_{12}, \tilde{A}_{22}, \tilde{A}_{23})' \quad (105)$$

Additionally, the variances of the estimates are deduced from the variance-covariance matrix

$$\begin{aligned} \text{Var}(\hat{\underline{a}}) &= (X'H^{-1}X)^{-1} \sigma_e^2 \\ &= \sigma_e^2 \frac{(1 + \hat{\gamma})(1 + 2\hat{\gamma})}{[(1 + 2\hat{\gamma}) + (1 + \hat{\gamma})^2]^2 - \hat{\gamma}^2(1 + \hat{\gamma})^2} \times \\ &\quad \begin{bmatrix} (1 + 2\hat{\gamma}) + (1 + \hat{\gamma})^2 & \hat{\gamma}(1 + \hat{\gamma}) \\ \hat{\gamma}(1 + \hat{\gamma}) & (1 + 2\hat{\gamma}) + (1 + \hat{\gamma})^2 \end{bmatrix} \end{aligned} \quad (106)$$

where $\hat{\gamma} = \hat{\sigma}_b^2 / \hat{\sigma}_e^2$, as estimated from the pilot data.

That means

$$\text{Var}(\hat{\alpha}_t) = \frac{(1 + \hat{\gamma})(1 + 2\hat{\gamma})[(1 + 2\hat{\gamma}) + (1 + \hat{\gamma})^2]}{[(1 + 2\hat{\gamma}) + (1 + \hat{\gamma})^2]^2 - \hat{\gamma}^2(1 + \hat{\gamma})^2} \sigma_e^2$$

for $t = 1, 2, T = 2$, or

$$\text{Var}(\hat{\alpha}_T) = \frac{(1 + \hat{\gamma})(1 + 2\hat{\gamma})[(1 + 2\hat{\gamma}) + (1 + \hat{\gamma})^2]}{[(1 + 2\hat{\gamma}) + (1 + \hat{\gamma})^2]^2 - \hat{\gamma}^2(1 + \hat{\gamma})^2} \sigma_e^2 \quad (107)$$

where $\hat{\gamma} = \hat{\sigma}_b^2 / \hat{\sigma}_e^2$, as estimated from the pilot data.

3.4.6.2 Case of $T = 3$

The rotation pattern ($S = 2, r = 3$) is presented below in figure 3-11.

		Year number		
Segment number	s \ t	1	2	3
	1	X		
	2	X	X	
	3		X	X
	4			X

Figure 3-11.- ($S = 2, r = \infty$) Patterns [2 retained 1; $T = 3$]

The design matrices related to the model shown as equation (12) are

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad (108)$$

and

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (109)$$

Then,

$$H = I + \gamma U U' = \begin{bmatrix} 1 + \gamma & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 + \gamma & \gamma & 0 & 0 & 0 \\ 0 & \gamma & 1 + \gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 + \gamma & \gamma & 0 \\ 0 & 0 & 0 & \gamma & 1 + \gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 + \gamma \end{bmatrix} \quad (110)$$

$$H^{-1} = \frac{1}{1 + 2\gamma} \begin{bmatrix} \frac{1 + 2\gamma}{1 + \gamma} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 + \gamma & -\gamma & 0 & 0 & 0 \\ 0 & -\gamma & 1 + \gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 + \gamma & -\gamma & 0 \\ 0 & 0 & 0 & -\gamma & 1 + \gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1 + 2\gamma}{1 + \gamma} \end{bmatrix} \quad (111)$$

and

$$X' H^{-1} X = \frac{1}{1 + 2\gamma} \begin{bmatrix} \frac{(1 + 2\gamma) + (1 + \gamma)^2}{1 + \gamma} & -\gamma & 0 \\ -\gamma & 2(1 + \gamma) & -\gamma \\ 0 & -\gamma & \frac{(1 + 2\gamma) + (1 + \gamma)^2}{1 + \gamma} \end{bmatrix} \quad (112)$$

So,

$$(X' H^{-1} X)^{-1} = (1 + 2\gamma) \begin{bmatrix} \frac{(1 + 2\gamma) + (1 + \gamma)^2}{1 + \gamma} & -\gamma & 0 \\ -\gamma & 2(1 + \gamma) & -\gamma \\ 0 & -\gamma & \frac{(1 + 2\gamma) + (1 + \gamma)^2}{1 + \gamma} \end{bmatrix}^{-1}$$

or, with $d \equiv (1 + 2\gamma) + (1 + \gamma)^2$, then

$$(X' H^{-1} X)^{-1} = (1 + 2\gamma) \begin{bmatrix} d/(1 + \gamma) & -\gamma & 0 \\ -\gamma & 2(1 + \gamma) & -\gamma \\ 0 & -\gamma & d/(1 + \gamma) \end{bmatrix}^{-1} \quad (113)$$

which can be written as

$$(X'H^{-1}X)^{-1} = (1 + 2\gamma) A^{-1} \quad (114)$$

where

$$|A| = \frac{2 d(d - \gamma^2)}{1 + \gamma} \quad (115)$$

and

$$A^{-1} = \frac{1 + \gamma}{2 d(d - \gamma^2)} \begin{bmatrix} 2 d - \gamma^2 & \frac{\gamma d}{1 + \gamma} & \gamma^2 \\ \frac{\gamma d}{1 + \gamma} & \frac{d^2}{(1 + \gamma)^2} & \frac{\gamma d}{1 + \gamma} \\ \gamma^2 & \frac{\gamma d}{1 + \gamma} & 2 d - \gamma^2 \end{bmatrix} \quad (116)$$

That means

$$(X'H^{-1}X)^{-1} = \frac{(1 + \gamma)(1 + 2\gamma)}{2 d(d - \gamma^2)} \begin{bmatrix} 2 d - \gamma^2 & \frac{\gamma}{1 + \gamma} d & \gamma^2 \\ \frac{\gamma}{1 + \gamma} d & \frac{d^2}{(1 + \gamma)^2} & \frac{\gamma}{1 + \gamma} d \\ \gamma^2 & \frac{\gamma}{1 + \gamma} d & 2 d - \gamma^2 \end{bmatrix} \quad (117)$$

Vector \underline{a} in the basic model shown in equation (12) for this rotation pattern is

$$\underline{a} = (A_{11}, A_{12}, A_{22}, A_{23}, A_{33}, A_{34})' \quad (118)$$

With the \tilde{A}_{ts} 's being the estimates of the A_{ts} 's of the above components (by step 4 of the flow chart in figure 3-10), the $\hat{\underline{a}}_t$'s of the estimate are derived from

$$\hat{\underline{a}} = (X'H^{-1}X)^{-1}X'H^{-1}\underline{a}^0 \quad (119)$$

where

$$\underline{a}^0 = (\tilde{A}_{11}, \tilde{A}_{12}, \tilde{A}_{22}, \tilde{A}_{23}, \tilde{A}_{33}, \tilde{A}_{34})' \quad (120)$$

And the variances of the estimates are deduced from the variance-covariance matrix as follows.

$$\begin{aligned}
\text{Var}(\hat{\alpha}) &= (X'H^{-1}X)^{-1}\sigma_e^2 \\
&= \sigma_e^2 \frac{(1 + \hat{\gamma})(1 + 2\hat{\gamma})}{2 \hat{d}(\hat{d} - \hat{\gamma}^2)} \\
&\quad \begin{bmatrix} 2 \hat{d} - \hat{\gamma}^2 & \frac{\hat{\gamma}}{1 + \hat{\gamma}} \hat{d} & \hat{\gamma}^2 \\ \frac{\hat{\gamma}}{1 + \hat{\gamma}} \hat{d} & \frac{\hat{\gamma}^2}{(1 + \hat{\gamma})^2} & \frac{\hat{\gamma}}{1 + \hat{\gamma}} \hat{d} \\ \hat{\gamma}^2 & \frac{\hat{\gamma}}{1 + \hat{\gamma}} \hat{d} & 2 \hat{d} - \hat{\gamma}^2 \end{bmatrix}
\end{aligned} \tag{121}$$

where $\hat{\gamma} = \hat{\sigma}_b^2 / \hat{\sigma}_e^2$, as estimated from the pilot data, and

$$\hat{d} = (1 + 2\hat{\gamma}) + (1 + \hat{\gamma})^2 \tag{122}$$

That means, for $T = 3$,

$$\boxed{\text{Var}(\hat{\alpha}_T) = \sigma_e^2 \frac{(1 + \hat{\gamma})(1 + 2\hat{\gamma})(2 \hat{d} - \hat{\gamma}^2)}{2 \hat{d}(\hat{d} - \hat{\gamma}^2)}} \tag{123}$$

where $\hat{d} \equiv (1 + 2\hat{\gamma}) + (1 + \hat{\gamma})^2$

and $\hat{\gamma} = \hat{\sigma}_b^2 / \hat{\sigma}_e^2$, as estimated from the pilot data.

3.4.6.3 Case of $T = 4$

The rotation pattern ($S = 2$, $r = \infty$; $T = 4$) is presented in figure 3-5, which is repeated as follows.

		Year number			
Segment number	s \ t	1	2	3	4
	1	X			
	2	X	X		
	3		X	X	
	4			X	X
	5				X

(S = 2, r = ∞) Patterns [2 retained 1; T = 4]

The design matrices related to the model, shown in equation (12) are:

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (124)$$

and

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (125)$$

Then, with $a = \gamma/(1 + \gamma)$, equation (125) can be written as

$$H = (1 + \gamma) \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & a & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (126)$$

hence,

$$H^{-1} = \frac{1}{1 + \gamma} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{1 - a^2} & \frac{-a}{1 - a^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-a}{1 - a^2} & \frac{1}{1 - a^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{1 - a^2} & \frac{-a}{1 - a^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-a}{1 - a^2} & \frac{1}{1 - a^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{1 - a^2} & \frac{-a}{1 - a^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-a}{1 - a^2} & \frac{1}{1 - a^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (127)$$

and

$$\begin{aligned}
 X'H^{-1}X &= \frac{1}{1+\gamma} \begin{bmatrix} 1 + \frac{1}{1-a^2} & \frac{-a}{1-a^2} & 0 & 0 \\ \frac{-a}{1-a^2} & \frac{2}{1-a^2} & \frac{-a}{1-a^2} & 0 \\ 0 & \frac{-a}{1-a^2} & \frac{2}{1-a^2} & \frac{-a}{1-a^2} \\ 0 & 0 & \frac{-a}{1-a^2} & 1 + \frac{1}{1-a^2} \end{bmatrix} \\
 &= \frac{1}{1+\gamma} \begin{bmatrix} f & g & 0 & 0 \\ g & 2h & g & 0 \\ 0 & g & 2h & g \\ 0 & 0 & g & f \end{bmatrix} \tag{128}
 \end{aligned}$$

where

$$f = 1 + \frac{1}{1-a^2} = [(1+2\gamma) + (1+\gamma)^2]/(1+2\gamma)$$

$$g = -a/(1+a^2) = -\gamma(1+\gamma)/(1+2\gamma)$$

$$h = 1/(1-a^2) = (1+\gamma)^2/(1+2\gamma)$$

By the Cramer rule in solving the system of equations,

$$\begin{cases}
 fx_1 + gx_2 & = 0 \\
 gx_1 + 2hx_2 + gx_3 & = 0 \\
 gx_2 + 2hx_3 + gx_4 & = 0 \\
 gx_3 + fx_4 & = 1
 \end{cases}$$

the value x_4 will give the following value of the element $(X'H^{-1}X)^{-1}_{44}$ as

$$(X'H^{-1}X)^{-1}_{44} = (1 + \gamma) \left(\begin{array}{ccc|c} f & g & 0 & 0 \\ g & 2h & g & 0 \\ 0 & g & 2h & 0 \\ \hline 0 & 0 & g & 1 \\ \hline f & g & 0 & 0 \\ g & 2h & g & 0 \\ 0 & g & 2h & g \\ \hline 0 & 0 & g & f \end{array} \right)$$

which, by being applied to the calculation of the determinant of a partitioned matrix, is

$$\begin{aligned} (X'H^{-1}X)^{-1}_{44} &= (1 + \gamma) \left\{ f - (0 \ 0 \ g) \begin{bmatrix} f & g & 0 \\ g & 2h & g \\ 0 & g & 2h \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} \right\}^{-1} \\ &= (1 + \gamma) \left\{ f - \frac{2fg^2h - g^4}{4fh - fg^2 - 2g^2h} \right\}^{-1} \\ &= (1 + \gamma) \frac{4fh - fg^2 - 2g^2h}{+4f^2h - f^2g^2 - 4fg^2h + g^4} \end{aligned}$$

i.e., for $T = 4$,

$$\text{Var}(\hat{\alpha}_T) = \sigma_e^2 (1 + \hat{\gamma}) (4f^0h - fg^{02} - 2g^{02}h) / (4f^{02}h - f^2g^{02} - 4fg^{02}h + g^4) \quad (129)$$

where,

$\hat{\gamma} = \hat{\sigma}_b^2 / \hat{\sigma}_e^2$, as estimated from the pilot data

$$f^0 = [(1 + 2\hat{\gamma}) + (1 + \hat{\gamma})^2] / (1 + 2\hat{\gamma})$$

$$g^0 = -\hat{\gamma}(1 + \hat{\gamma}) / (1 + 2\hat{\gamma})$$

$$\hat{h}^0 = (1 + \hat{\gamma})^2 / (1 + 2\hat{\gamma})$$

3.4.7 RESOLUTION FOR ($S = 3, r = \infty$) ROTATION PATTERNS

3.4.7.1 Case of $T = 3$

The rotation pattern ($S = 3, r = \infty; T = 3$) is presented below in figure 3-12.

		Year number		
Segment number	s \ t	1	2	3
	1	X		
	2	X	X	
	3	X	X	X
	4		X	X
	5			X

Figure 3-12 - ($S = 3; r = \infty$) Patterns [3 retained 2; $T = 3$].
While this pattern shows the notation for only three years,
the related pattern given in figure 3-6 shows it for four.

The design matrices related to the basic model, shown in equation 12, are

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad (130)$$

and

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (131)$$

Then, with $a = \gamma(1 + \gamma)$, $U = (1 - a)(1 + 2a) = (1 + 3\gamma)/(1 + \gamma)^2$

$$H = (1 + \gamma) \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & a & a & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 1 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & a & a & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (132)$$

$$H^{-1} = \frac{1}{1 + \gamma} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{1 - a^2} & \frac{-a}{1 - a^2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-a}{1 - a^2} & \frac{1}{1 - a^2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1 + a}{U} & \frac{-a}{U} & \frac{-a}{U} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-a}{U} & \frac{1 + a}{U} & \frac{-a}{U} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-a}{U} & \frac{-a}{U} & \frac{1 + a}{U} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{1 - a^2} & \frac{-a}{1 - a^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-a}{1 - a^2} & \frac{1}{1 - a^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (133)$$

So,

$$X'H^{-1}X = \frac{1}{1+\gamma} \begin{bmatrix} 1 + \frac{1}{1-a^2} + \frac{1+a}{u} & \frac{-a}{u} - \frac{a}{1-a^2} & \frac{-a}{u} \\ \frac{a}{u} - \frac{a}{1-a^2} & 2\frac{1}{1-a^2} + \frac{1+a}{u} & \frac{-a}{u} - \frac{a}{1-a^2} \\ \frac{-a}{u} & \frac{-a}{u} - \frac{a}{1-a^2} & 1 + \frac{1}{1-a^2} + \frac{1+a}{u} \end{bmatrix} \quad (134)$$

which can be written

$$X'H^{-1}X = \frac{1}{1+\gamma} \begin{bmatrix} A & B & D \\ B & C & B \\ D & B & A \end{bmatrix} \quad (135)$$

where

$$A \equiv 1 + \frac{1}{1-a^2} + \frac{1+a}{u} = \frac{(1+2\gamma)(1+3\gamma) + (1+\gamma)^2(1+3\gamma) + (1+\gamma)(1+2\gamma)^2}{(1+2\gamma)(1+3\gamma)} \quad (136)$$

$$B \equiv \frac{-a}{u} - \frac{a}{1-a^2} = \frac{-\gamma(1+\gamma)[(1+2\gamma) + (1+3\gamma)]}{(1+2\gamma)(1+3\gamma)} \quad (137)$$

$$C \equiv 2\frac{1}{1-a^2} + \frac{1+a}{u} = \frac{(1+\gamma)[2(1+\gamma)(1+3\gamma) + (1+2\gamma)^2]}{(1+2\gamma)(1+3\gamma)} \quad (138)$$

and

$$D \equiv \frac{-a}{u} = \frac{-\gamma(1+\gamma)(1+2\gamma)}{(1+2\gamma)(1+3\gamma)} \quad (139)$$

In applying the definition of the inversion of a matrix, the three diagonal elements of the inverse of $X'H^{-1}X$ will be

$$(X'H^{-1}X)_{11}^{-1} = (X'H^{-1}X)_{33}^{-1} = (1+\gamma) \frac{AC - B^2}{(A-D)[(A+D)C - 2B^2]} \quad (140)$$

and

$$(X'H^{-1}X)_{22}^{-1} = (1+\gamma) \frac{A+D}{A+D-B^2} \quad (141)$$

That means

$$\text{Var } \hat{\alpha}_T = (1 + \hat{\gamma}) \frac{\begin{matrix} 00 \\ (AC - B^2) \end{matrix}}{\begin{matrix} 00 & 00 \\ (A - D) & [(A + D)C - 2B^2] \end{matrix}} \sigma_e^2 \quad (142)$$

for $T = 3$, where

$\hat{\gamma} = \hat{\sigma}_b^2 / \hat{\sigma}_e^2$, as estimated from pilot data

$$\overset{0}{A} = \frac{(1 + 2\hat{\gamma})(1 + 3\hat{\gamma}) + (1 + \hat{\gamma})^2(1 + 3\hat{\gamma}) + (1 + \hat{\gamma})(1 + 2\hat{\gamma})^2}{(1 + 2\hat{\gamma})(1 + 3\hat{\gamma})} \quad (143)$$

$$\overset{0}{B} = \frac{-\hat{\gamma}(1 + \hat{\gamma})[(1 + 2\hat{\gamma}) + (1 + 3\hat{\gamma})]}{(1 + 2\hat{\gamma})(1 + 3\hat{\gamma})} \quad (144)$$

$$\overset{0}{C} = \frac{(1 + \hat{\gamma})[2(1 + \hat{\gamma})(1 + 3\hat{\gamma}) + (1 + 2\hat{\gamma})^2]}{(1 + 2\hat{\gamma})(1 + 3\hat{\gamma})} \quad (145)$$

and

$$\overset{0}{D} = \frac{-\hat{\gamma}(1 + \hat{\gamma})(1 + 2\hat{\gamma})}{(1 + 2\hat{\gamma})(1 + 3\hat{\gamma})} \quad (146)$$

Note: In applying the Cramer approach to directly solve for x_3 in the system of three equations

$$\begin{cases} Ax_1 + Bx_2 + Dx_3 = 0 \\ Bx_1 + Cx_2 + Bx_3 = 0 \\ Dx_1 + Bx_2 + Ax_3 = 1 \end{cases} \quad (147)$$

the value of the element $(X'H^{-1}X)_{33}^{-1}$ will be as follows.

$$(X'H^{-1}X)_{33}^{-1} = (1 + \gamma) \cdot \begin{array}{|cc|c|} \hline A & B & 0 \\ \hline B & C & 0 \\ \hline D & B & 1 \\ \hline A & B & D \\ \hline B & C & B \\ \hline D & B & A \\ \hline \end{array} \quad (148)$$

and by calculating the determinant of a matrix that is partitioned,

that is $(X'H^{-1}X)_{33}^{-1} = (1 + \gamma) \left\{ \left| A - (DB) \begin{pmatrix} A & B \\ B & C \end{pmatrix}^{-1} \begin{pmatrix} D \\ B \end{pmatrix} \right| \right\}^{-1}$ (149)

$$\begin{aligned} (X'H^{-1}X)_{33}^{-1} &= (1 + \gamma) \frac{AC - B^2}{|A(AC - B^2) - D^2C + 2B^2D - AB^2|} \\ &= (1 + \gamma) \frac{AC - B^2}{A^2C - 2AB^2 - D^2C + 2B^2D} \\ &= (1 + \gamma) \frac{AC - B^2}{(A^2 - D^2)C - 2B^2(A - D)} \\ &= (1 + \gamma) \frac{AC - B^2}{(A - D)[(A + D)C - 2B^2]} \end{aligned} \quad (150)$$

which is the same as equation (140).

3.4.7.2 Case of $T = 4$

The rotation pattern ($S = 3$, $r = \infty$; $T = 4$) is presented in figure 3-6, which is repeated as follows.

		Year number			
Segment number	s \ t	1	2	3	4
	1	X			
	2	X	X		
	3	X	X	X	
	4		X	X	X
	5			X	X
	6				X

(S = 3, r = ∞) Patterns [3 retained 2; T = 4]

The design matrices related to the basic model, as shown on equation (12), are

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (151)$$

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (152)$$

Then, with $a = \gamma/(1 + \gamma)$, $u = (1 - a)(1 + 2a) = (1 + 3\gamma)/(1 + \gamma)^2$,

$$H = (1 + \gamma) \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & a & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 1 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & a & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & a & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & 1 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & a & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (153)$$

$$H^{-1} = \frac{1}{1+\gamma} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{1-a^2} & \frac{-a}{1-a^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-a}{1-a^2} & \frac{1}{1-a^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1+a}{u} & \frac{-a}{u} & \frac{-a}{u} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-a}{u} & \frac{1+a}{u} & \frac{-a}{u} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-a}{u} & \frac{-a}{u} & \frac{1+a}{u} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+a}{u} & \frac{-a}{u} & \frac{-a}{u} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-a}{u} & \frac{1+a}{u} & \frac{-a}{u} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-a}{u} & \frac{-a}{u} & \frac{1+a}{u} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{1-a^2} & \frac{-a}{1-a^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-a}{1-a^2} & \frac{1}{1-a^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (154)$$

and,

$$X'H^{-1}X = \frac{1}{1+\gamma} \begin{bmatrix} 1 + \frac{1}{1-a^2} + \frac{1+a}{u} & -\frac{a}{1-a^2} - \frac{a}{u} & -\frac{a}{u} & 0 \\ -\frac{a}{1-a^2} - \frac{a}{u} & \frac{1}{1-a^2} + 2\frac{1+a}{u} & -2\frac{a}{u} & -\frac{a}{u} \\ -\frac{a}{u} & -2\frac{a}{u} & \frac{1}{1-a^2} - 2\frac{1+a}{u} & -\frac{a}{1-a^2} - \frac{a}{u} \\ 0 & -\frac{a}{u} & -\frac{a}{1-a^2} - \frac{a}{u} & 1 + \frac{1}{1-a^2} + \frac{1+a}{u} \end{bmatrix}$$

$$= \frac{1}{1+\gamma} \begin{bmatrix} A & B & D & 0 \\ B & E & 2D & D \\ D & 2D & E & B \\ 0 & D & B & A \end{bmatrix} \quad (155)$$

where A, B, D are defined in equations (137), (138), and (139), and

$$E = \frac{1}{1-a^2} + 2\frac{1+a}{u} = \frac{(1+\gamma)[(1+\gamma)(1+3\gamma) + 2(1+2\gamma)^2]}{(1+2\gamma)(1+3\gamma)}$$

Therefore,

$$(X'H^{-1}X)_{44}^{-1} = (1 + \gamma) \left\{ \begin{array}{ccc|c} A & B & D & 0 \\ B & E & 2D & 0 \\ D & 2D & E & 0 \\ \hline 0 & D & B & 1 \\ \hline A & B & D & 0 \\ B & E & 2D & D \\ D & 2D & E & B \\ \hline 0 & D & B & A \end{array} \right\}^{-1} \left\{ \begin{array}{c} 0 \\ D \\ B \end{array} \right\}^{-1} \quad (156)$$

Hence, for $T = 4$,

$$\text{Var } \hat{\alpha}_T = (1 + \hat{\gamma}) \left\{ \begin{array}{c} 0 \\ A - \begin{pmatrix} 0 & 0 & 1 \\ \hat{b}' & \hat{M} & \hat{b} \end{pmatrix}^{-1} \end{array} \right\}^{-1} \sigma_e^2 \quad (157)$$

where

$$m = (1 + 3\hat{\gamma}) / (1 + \hat{\gamma})^3$$

$$\begin{matrix} 0 \\ A \end{matrix} = \frac{[(1 + 2\hat{\gamma})(1 + 3\hat{\gamma}) + (1 + \hat{\gamma})^2(1 + 3\hat{\gamma}) + (1 + \hat{\gamma})(1 + 2\hat{\gamma})^2]}{(1 + 2\hat{\gamma})(1 + 3\hat{\gamma})} \quad (158)$$

$$\begin{matrix} 0 \\ B \end{matrix} = \frac{-\hat{\gamma}(1 + \hat{\gamma})[(1 + 2\hat{\gamma}) + (1 + 3\hat{\gamma})]}{(1 + 2\hat{\gamma})(1 + 3\hat{\gamma})} \quad (159)$$

$$\begin{matrix} 0 \\ D \end{matrix} = \frac{-\hat{\gamma}(1 + \hat{\gamma})(1 + 2\hat{\gamma})}{(1 + 2\hat{\gamma})(1 + 3\hat{\gamma})} \quad (160)$$

$$\begin{matrix} 0 \\ E \end{matrix} = \frac{(1 + \hat{\gamma})[(1 + \hat{\gamma})(1 + 3\hat{\gamma}) + 2(1 + 2\hat{\gamma})^2]}{(1 + 2\hat{\gamma})(1 + 3\hat{\gamma})} \quad (161)$$

$${}^0M = \begin{bmatrix} {}^0A & {}^0B & {}^0D \\ {}^0B & {}^0E & 2{}^0D \\ {}^0D & 2{}^0D & {}^0E \end{bmatrix} \quad (162)$$

and

$${}^0b' = [{}^0D \quad {}^0D \quad {}^0B] \quad (163)$$

3.4.8 RESOLUTION FOR (S = 4, r = ∞) ROTATION PATTERNS

3.4.8.1 Case of T = 3

The rotation pattern (S = 4, r = ∞; T = 3) is presented below in figure 3-13.

		Year number		
Segment number	s \ t	1	2	3
	1	X		
	2	X	X	
	3	X	X	X
	4	X	X	X
	5		X	X
	6			X

Figure 3-13.- (S = 4, r = ∞) Pattern [4 retained 3; T = 3].

The design matrices related to the basic equation (12) are

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad (164)$$

and

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (165)$$

Then, with $a = \gamma/(1 + \gamma)$,

$$H = (1 + \gamma) \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & a & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 1 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & a & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & a & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & 1 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & a & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (166)$$

and by applying the results given in appendix A with

$$\begin{aligned} u &= (1 - a)(1 + 2a) \\ &= (1 + 3\gamma)/(1 + \gamma)^2 \end{aligned}$$

then

$$H^{-1} = \frac{1}{1 + \gamma} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{1 - a^2} & \frac{-a}{1 - a^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-a}{1 - a^2} & \frac{1}{1 - a^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1 + a}{u} & \frac{-a}{u} & \frac{-a}{u} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-a}{u} & \frac{1 + a}{u} & \frac{-a}{u} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-a}{u} & \frac{-a}{u} & \frac{1 + a}{u} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1 + a}{u} & \frac{-a}{u} & \frac{-a}{u} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-a}{u} & \frac{1 + a}{u} & \frac{-a}{u} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-a}{u} & \frac{-a}{u} & \frac{1 + a}{u} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{1 - a^2} & \frac{-a}{1 - a^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-a}{1 - a^2} & \frac{1}{1 - a^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (167)$$

and

$$\begin{aligned}
 X'H^{-1}X &= \frac{1}{1+\gamma} \begin{bmatrix} 1 + \frac{1}{1-a^2} + 2\frac{1+a}{u} & -\frac{a}{1-a^2} - 2\frac{a}{u} & -2\frac{a}{u} \\ -\frac{a}{1-a^2} - 2\frac{a}{u} & 2\left(\frac{1}{1-a^2} + \frac{1+a}{u}\right) & -\frac{a}{1-a^2} - 2\frac{a}{u} \\ -2\frac{a}{u} & -\frac{a}{1-a^2} - 2\frac{a}{u} & 1 + \frac{1}{1-a^2} + 2\frac{1+a}{u} \end{bmatrix} \\
 &= \frac{1}{1+\gamma} \begin{bmatrix} F & G & 2D \\ G & H & G \\ 2D & G & F \end{bmatrix} \quad (168)
 \end{aligned}$$

where

$$F \equiv 1 + \frac{1}{1-a^2} + 2\frac{1+a}{(1-a)(1+2a)} = \frac{(1+2\gamma)(1+3\gamma) + (1+\gamma)^2(1+3\gamma) + 2(1+\gamma)(1+2\gamma)^2}{(1+2\gamma)(1+3\gamma)} \quad (169)$$

$$G \equiv -\frac{a}{1-a^2} - 2\frac{a}{(1-a)(1+2a)} = \frac{-\gamma(1+2\gamma)[2(1+2\gamma) + (1+3\gamma)]}{(1+2\gamma)(1+3\gamma)} \quad (170)$$

$$H \equiv 2\left(\frac{1}{1-a^2} + \frac{1+a}{(1-a)(1+2a)}\right) = \frac{2(1+\gamma)[(1+\gamma)(1+3\gamma) + (1+2\gamma)^2]}{(1+2\gamma)(1+3\gamma)} \quad (171)$$

$$D \equiv -\frac{a}{u} = -\frac{\gamma(1+\gamma)(1+2\gamma)}{(1+2\gamma)(1+3\gamma)} \text{ as in equation (140).} \quad (172)$$

Therefore,

$$\begin{aligned}
 (X'H^{-1}X)^{-1}_{33} &= (1+\gamma) \begin{vmatrix} F & G & 0 \\ G & H & 0 \\ 2D & G & 1 \end{vmatrix} \\
 &= (1+\gamma) \left\{ |F \ G| \begin{pmatrix} F & G \\ G & H \end{pmatrix}^{-1} \begin{pmatrix} 2D \\ G \end{pmatrix} \right\} \\
 &= (1+\gamma) \frac{(FH - G^2)}{F(FH - G^2) - 4D^2H + 4DG^2 - FG^2} \\
 &= (1+\gamma) \frac{FH - G^2}{(F - 2D)[(F + 2D)H - 2G^2]} \quad (173)
 \end{aligned}$$

In fact,

$$(X'H^{-1}X)_{11}^{-1} = (X'H^{-1}X)_{33}^{-1}$$

Hence, for $T = 3$,

$$\text{Var } \hat{\alpha}_T = (1 + \hat{\gamma}) \frac{\begin{matrix} 00 \\ (FH - G^2) \end{matrix}}{\begin{matrix} 0 & 0 & 0 \\ (F - 2D) & [(F + 2D)H - 2G^2] \end{matrix}} \quad (174)$$

where

$$\hat{\gamma} = \hat{\sigma}_b^2 / \hat{\sigma}_e^2, \text{ as estimated from the pilot data}$$

$$\begin{matrix} 0 \\ F \end{matrix} = \frac{[(1 + 2\hat{\gamma})(1 + 3\hat{\gamma}) + (1 + \hat{\gamma})^2(1 + 3\hat{\gamma}) + 2(1 + \hat{\gamma})(1 + 2\hat{\gamma})^2]}{(1 + 2\hat{\gamma})(1 + 3\hat{\gamma})} \quad (175)$$

$$\begin{matrix} 0 \\ G \end{matrix} = \frac{-\hat{\gamma}(1 + 2\hat{\gamma})[2(1 + 2\hat{\gamma}) + (1 + 3\hat{\gamma})]}{(1 + 2\hat{\gamma})(1 + 3\hat{\gamma})} \quad (176)$$

$$\begin{matrix} 0 \\ H \end{matrix} = \frac{2(1 + \hat{\gamma})[(1 + \hat{\gamma})(1 + 3\hat{\gamma}) + (1 + 2\hat{\gamma})^2]}{(1 + 2\hat{\gamma})(1 + 3\hat{\gamma})} \quad (177)$$

$$\begin{matrix} 0 \\ D \end{matrix} = \frac{-\hat{\gamma}(1 + \hat{\gamma})(1 + 2\hat{\gamma})}{(1 + 2\hat{\gamma})(1 + 3\hat{\gamma})} \quad (178)$$

3.4.8.2 Case of $T = 4$

The rotation pattern ($S = 4$, $r = \infty$; $T = 4$) is presented in figure 3-7, which is repeated below.

Year number				
s \ t	1	2	3	4
1	X			
2	X	X		
3	X	X	X	
4	X	X	X	X
5		X	X	X
6			X	X
7				X

Figure 3-7.- ($S = 4$, $r = \infty$) Patterns [4 retained 3].

The design matrices related to the based model as shown in equation (12) are

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (179)$$

and

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (180)$$

Then, with $a = \gamma/1 + \gamma$,

$$H = (1 + \gamma) \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & a & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 1 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & a & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & a & a & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & 1 & a & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & a & 1 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & a & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 1 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & a & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (181)$$

and, by applying the results given in appendix A, we denote equations (182) through (184)

$$\left. \begin{aligned} \eta &\equiv (1 - a)(1 + 3a) = (1 + 4\gamma)/(1 + \gamma)^2 \\ u &\equiv (1 - a)(1 + 2a) = (1 + 3\gamma)/(1 + \gamma)^2 \end{aligned} \right\} \quad (182)$$

where

$$J \equiv 1 + \frac{1}{1 - a^2} + \frac{1 - a}{u} + \frac{1 + 2a}{\eta} = 1 + \frac{(1 + \gamma)^2}{1 + 2\gamma} + \frac{1 + \gamma}{1 + 3\gamma} + \frac{(1 + \gamma)(1 + 3\gamma)}{1 + 4\gamma} \quad (185)$$

$$K \equiv -\frac{a}{1 - a^2} - \frac{a}{u} - \frac{a}{\eta} = -\frac{\gamma(1 + \gamma)}{(1 + 2\gamma)} - \frac{\gamma(1 + \gamma)}{1 + 3\gamma} - \frac{\gamma(1 + \gamma)}{1 + 4\gamma} \quad (186)$$

$$L \equiv -\frac{a}{u} - \frac{a}{\eta} = -\frac{\gamma(1 + \gamma)}{1 + 3\gamma} - \frac{\gamma(1 + \gamma)}{1 + 4\gamma} \quad (187)$$

$$R \equiv -\frac{a}{\eta} = -\frac{\gamma(1 + \gamma)}{1 + 4\gamma} \quad (188)$$

(183)

[illegible]

$$X'H^{-1}X = \frac{1}{1+\gamma} \begin{bmatrix} 1 + \frac{1}{1-a^2} + \frac{1-a}{u} + \frac{1+2a}{n} & -\frac{a}{1-a^2} - \frac{a}{u} - \frac{a}{n} & -\frac{a}{u} - \frac{a}{n} & -\frac{a}{n} \\ -\frac{a}{1-a^2} - \frac{a}{u} - \frac{a}{n} & \frac{1}{1-a^2} + 2\frac{1+a}{u} + \frac{1+2a}{n} & -\frac{a}{u} - \frac{a}{n} & -\frac{a}{u} - \frac{a}{n} \\ -\frac{a}{u} - \frac{a}{n} & -\frac{a}{u} - \frac{a}{n} & 2\frac{1+a}{u} + \frac{1+2a}{n} + \frac{1}{1-a^2} & -\frac{a}{u} - \frac{a}{n} - \frac{a}{1-a^2} \\ -\frac{a}{n} & -\frac{a}{u} - \frac{a}{n} & -\frac{a}{u} - \frac{a}{n} - \frac{a}{1-a^2} & 1 + \frac{1}{1-a^2} + \frac{1-a}{u} + \frac{1+2a}{n} \end{bmatrix}$$

or

$$X'H^{-1}X = \frac{1}{1+\gamma} \begin{bmatrix} J & K & L & R \\ K & P & Q & L \\ L & Q & P & K \\ R & L & K & J \end{bmatrix}$$

(134)

$$Q = -2\frac{a}{u} - \frac{a}{n} = -\frac{2\gamma(1+\gamma)}{1+3\gamma} - \frac{\gamma(1+\gamma)}{1+4\gamma} \quad (189)$$

$$P = \frac{1}{1-a^2} + 2\frac{1+a}{u} + \frac{1+2a}{n} = \frac{(1+\gamma)^2}{1+2\gamma} + \frac{2(1+\gamma)(1+2\gamma)}{1+3\gamma} + \frac{(1+\gamma)(1+3\gamma)}{1+4\gamma} \quad (190)$$

Therefore,

$$(X'H^{-1}X)_{44}^{-1} = (1+\gamma) \left(\begin{array}{ccc|c} J & K & L & 0 \\ K & P & Q & 0 \\ L & Q & P & 0 \\ \hline R & L & K & 1 \end{array} \right) \quad (191)$$

$$= (1+\gamma) \left\{ J - (R \ L \ K) \begin{bmatrix} J & K & L \\ K & P & Q \\ L & Q & P \end{bmatrix}^{-1} \begin{bmatrix} R \\ L \\ K \end{bmatrix} \right\}^{-1}$$

Hence, for $T = 4$,

$$\text{Var } \hat{\alpha}_T = (1 + \hat{\gamma}) \left\{ \begin{array}{ccc} 0 & 0 & 0 \\ J & \hat{\gamma} & N^{-1} \\ 0 & \hat{\gamma} & 0 \end{array} \right\}^{-1} \sigma_e^2 \quad (192)$$

where

$\hat{\gamma} = \hat{\sigma}_b^2 / \hat{\sigma}_e^2$, as estimated from the pilot data

$$\hat{J} = 1 + \frac{(1 + \hat{\gamma})^2}{1 + 2\hat{\gamma}} + \frac{1 + \hat{\gamma}}{1 + 3\hat{\gamma}} + \frac{(1 + \hat{\gamma})(1 + 3\hat{\gamma})}{1 + 4\hat{\gamma}} \quad (193)$$

$${}^0_K = -\frac{\hat{\gamma}(1 + \hat{\gamma})}{1 + 2\hat{\gamma}} - \frac{\hat{\gamma}(1 + \hat{\gamma})}{1 + 3\hat{\gamma}} - \frac{\hat{\gamma}(1 + \hat{\gamma})}{1 + 4\hat{\gamma}} \quad (194)$$

$${}^0_R = -\hat{\gamma}(1 + \hat{\gamma})/(1 + 4\hat{\gamma}) \quad (195)$$

$${}^0_L = -\hat{\gamma}(1 + \hat{\gamma})/(1 + 3\hat{\gamma}) - \hat{\gamma}(1 + \hat{\gamma})/(1 + 4\hat{\gamma}) \quad (196)$$

$${}^0_Q = -2\hat{\gamma}(1 + \hat{\gamma})/(1 + 3\hat{\gamma}) - \hat{\gamma}(1 + \hat{\gamma})/(1 + 4\hat{\gamma}) \quad (197)$$

$${}^0_P = (1 + \hat{\gamma})^2/(1 + 2\hat{\gamma}) + 2(1 + \hat{\gamma})(1 + 2\hat{\gamma})/(1 + 3\hat{\gamma}) + (1 + \hat{\gamma})(1 + 3\hat{\gamma})/(1 + 4\hat{\gamma}) \quad (198)$$

$${}^0_{\tilde{Y}'} = ({}^0_R \ {}^0_L \ {}^0_K) \quad (199)$$

and

$${}^0_N = \begin{bmatrix} {}^0_J & {}^0_K & {}^0_L \\ {}^0_K & {}^0_P & {}^0_Q \\ {}^0_L & {}^0_Q & {}^0_P \end{bmatrix} \quad (200)$$

3.4.9 RESOLUTIONS FOR OTHER SPECIFIED ROTATION PATTERNS

Following are some other specified rotation patterns of $S = 4$ segments per year, the results of which are to be presented and applied in the cases of necessary collapsions mentioned in section 3.3.2.

3.4.9.1 Resolution for ($S = 4, r = 3$) Rotation Patterns

The rotation pattern ($S = 4, r = 3$; [4 retained 3]) is presented in figure 3-14.

Segment number	Year number						
	<div>s \ t</div>	1	2	3	4	5	6
	1	X			X	X	X
	2	X	X			X	X
	3	X	X	X			X
	4	X	X	X	X		
	5		X	X	X	X	
	6			X	X	X	X

Figure 3-14.- (S = 4, r = 3) Patterns [4 retained 3].

Some results are

$$X'H^{-1}X = \frac{1}{1 + 4\gamma} \text{Circ}[4(1 + 3\gamma), -3\gamma, -2\gamma, -2\gamma, -2\gamma, -3\gamma] \quad (201)$$

$$(X'H^{-1}X)^{-1}_{66} = \frac{1 + 4\gamma}{6} \left(\frac{1}{4} + \frac{1}{4 + 16\gamma} - \frac{\sqrt{3}}{4 + 14\gamma - \sqrt{3}\gamma} - \frac{\sqrt{3}}{4 + 10\gamma + 5\sqrt{3}\gamma} \right) \quad (202)$$

and, for T = 6,

$$\text{Var}(\hat{\alpha}_T) = \frac{1 + 4\hat{\gamma}}{6} \left(\frac{1}{4} + \frac{1}{4 + 16\hat{\gamma}} - \frac{\sqrt{3}}{4 + 16\hat{\gamma} - \sqrt{3}\hat{\gamma}} - \frac{\sqrt{3}}{4 + 10\hat{\gamma} + 5\sqrt{3}\hat{\gamma}} \right) \sigma_e^2 \quad (203)$$

where $\hat{\gamma} = \hat{\sigma}_b^2 / \hat{\sigma}_e^2$, as estimated from the pilot data.

3.4.9.2 Resolution for (S = 4, r = 4) Rotation Patterns

The rotation pattern (S = 4, r = 4; [4 retained 3]) is presented in figure 3-15.

		Year number						
Segment number	s \ t	1	2	3	4	5	6	7
	1	X				X	X	X
	2	X	X				X	X
	3	X	X	X				X
	4	X	X	X	X			
	5		X	X	X	X		
	6			X	X	X	X	
	7				X	X	X	X

Figure 3-15.- (S = 4, r = 4) Patterns [4 retained 3].

Some results are

$$X'H^{-1}X = \frac{1}{1+4\gamma} \text{Circ}[4(1+3\gamma), -3\gamma, -2\gamma, -\gamma, -\gamma, -2\gamma, -2\gamma, -3\gamma] \quad (204)$$

$$(X'H^{-1}X)^{-1}_{77} = \frac{1+4\gamma}{7} \left(\frac{1}{4} + \frac{1.470}{4+15.357\gamma} - \frac{1.802}{4+15.602\gamma} - \frac{0.445}{4+10.951\gamma} \right) \quad (205)$$

and, for T = 7,

$$\text{Var } \hat{\alpha}_T = \frac{1+4\hat{\gamma}}{7} \left(\frac{1}{4} + \frac{1.470}{4+15.357\hat{\gamma}} - \frac{1.802}{4+15.602\hat{\gamma}} - \frac{0.445}{4+10.951\hat{\gamma}} \right) \quad (206)$$

where $\hat{\gamma} = \hat{\sigma}_h^2 / \hat{\sigma}_e^2$, as estimated from the pilot data.

3.4.9.3 Resolution for (S = 4, r = 2) Rotation Patterns

The rotation pattern (S = 4, r = 2; [4 retained 2]) is presented in figure 3-16.

		Year number		
Segment number	s \ t	1	2	3
	.1	X		X
	2	X		X
	3	X	X	
	4	X	X	
	5		X	X
	6		X	X

Figure 3-16.- (S = 4, r = 2) Patterns [4 retained 2].

Some results are

$$\begin{aligned}
 X'H^{-1}X &= \frac{2}{1+2\gamma} \begin{bmatrix} 2(1+\gamma) & -\gamma & -\gamma \\ -\gamma & 2(1+\gamma) & -\gamma \\ -\gamma & -\gamma & 2(1+\gamma) \end{bmatrix} \\
 &= 2(X'H^{-1}X)_{\text{Fig. 3-1}}
 \end{aligned} \tag{207}$$

$$(X'H^{-1}X)^{-1} = \frac{1}{2}(X'H^{-1}X)^{-1}_{\text{Fig 3-1}} \tag{208}$$

and, for T = 3,

$$\boxed{\text{Var}(\hat{\alpha}_T) = \frac{1}{2} \text{Var}(\hat{\alpha}_T)_{\text{Fig. 1}} = \frac{(1+2\hat{\gamma})(2+\hat{\gamma})}{4(2+3\hat{\gamma})} \sigma_e^2} \tag{209}$$

3.4.9.4 Resolution for (S = 4, r = 3) [4 retained 2] Rotation Patterns

The rotation pattern (S = 4, r = 3; [4 retained 2]) is presented in figure 3-17.

		Year number			
Segment number	s \ t	1	2	3	4
	1	X			X
	2	X			X
	3	X	X		
	4	X	X		
	5		X	X	
	6		X	X	
	7			X	X
	8			X	X

Figure 3-17.- (S = 4, r = 3) Pattern [4 retained 2].

Some results are

$$\begin{aligned}
 X'H^{-1}X &= \frac{2}{1+2\gamma} \text{Circ}[2(1+\gamma), -\gamma, 0, -\gamma] \\
 &= 2(X'H^{-1}X)_{\text{Fig. 3}}
 \end{aligned} \tag{210}$$

$$(X'H^{-1}X)^{-1} = \frac{1}{2}(X'H^{-1}X)^{-1}_{\text{Fig. 3}} \tag{211}$$

and, for T = 4,

$$\text{Var}(\hat{\alpha}_T) = \frac{1}{2} \text{Var}(\hat{\alpha}_T)_{\text{Fig. 3}} = \frac{1+2\hat{\gamma}}{16} \left(1 + \frac{1}{1+2\hat{\gamma}} - \frac{2}{1+\hat{\gamma}} \right) \sigma_e^2 \tag{212}$$

3.4.9.5 Resolution for (S = 4, r = ∞) Rotation Patterns [4 Retained 2]

The rotation pattern (S = 4, r = ∞; T = 3, [4 retained 2]) is presented in figure 3-18.

		Year number		
Segment number	s \ t	1	2	3
	1	X		
	2	X		
	3	X	X	
	4	X	X	
	5		X	X
	6		X	X
	7			X
	8			X

Figure 3-18.- (S = 4, r = ∞) Patterns [4 retained 2; T = 3].

Some results are

$$f \equiv 1 + 1/(1 - a^2) = [(1 + 2\gamma) + (1 + \gamma)^2]/(1 + 2\gamma)$$

$$g \equiv -a/(1 - a^2) = -\gamma(1 + \gamma)/(1 + 2\gamma)$$

$$h \equiv 1/(1 - a^2) = (1 + \gamma)^2/(1 + 2\gamma), \text{ as in equation (128)}$$

$$X'H^{-1}X = \frac{2}{1 + \gamma} \begin{bmatrix} f & g & 0 \\ g & 2h & g \\ 0 & g & f \end{bmatrix} \quad (213)$$

$$(X'H^{-1}X)^{-1} = \frac{1 + \gamma}{2} \frac{1}{2f(fh - g^2)} \begin{bmatrix} 2fh - g^2 & -gf & g^2 \\ -gf & f^2 & -gf \\ g^2 & -gf & 2fh - g^2 \end{bmatrix} \quad (214)$$

and, for T = 3,

$$\text{Var}(\hat{\alpha}_T) = \sigma_e^2 (1 + \hat{\gamma})(2fh - g^2)/4f(fh - g^2) \quad (215)$$

where

$$\left. \begin{aligned} f^0 &= [(1 + 2\hat{\gamma}) + (1 + \hat{\gamma})^2]/(1 + 2\hat{\gamma}) \\ g^0 &= -\hat{\gamma}(1 + \hat{\gamma})/(1 + 2\hat{\gamma}) \\ h^0 &= (1 + \hat{\gamma})^2/(1 + 2\hat{\gamma}) \end{aligned} \right\} \quad (216)$$

3.5 EMPIRICAL OPTIMIZATION, WITH PRECISION ACHIEVABLE BY ROTATION SAMPLE DESIGN ESTIMATION

As frequently mentioned in previous sections, the estimated values $\hat{\gamma}$ of γ are obtained from Landsat (pilot) data. In all numerical evaluations of the variance formulas, these estimated values $\hat{\gamma}$ are to be substituted.

In addition, as discussed in section 1.5.1, the objective of rotation sample designs is to utilize the "consistency" of the wheat acreage of a particular segment from year to year in order to improve the accuracy of the current year's wheat acreage estimate. This accuracy may be seen from the gain in precision which is attainable compared with the aggregation based only on the current year's estimates.

The statistic presenting the gain in precision is named the variance reduction ratio R.

3.5.1 VARIANCE-REDUCTION RATIO R

3.5.1.1 Variance $V(\bar{Y}_T)$ of Estimators Based Only on Current Year Acquisitions

For any value of T, for example $T = 3$, the estimator \bar{Y}_T can be shown as follows.

$$\bar{Y}_T = \frac{1}{S} (0 \dots 0 | 0 \dots 0 | 1 \dots 1 | 0 \dots 0 | 0 \dots 0) \begin{bmatrix} A_{11} \\ \vdots \\ A_{1S} \\ \vdots \\ A_{T1} \\ \vdots \\ A_{TS} \\ \vdots \end{bmatrix}$$

$$= \frac{1}{S} \bar{h}' \bar{a}$$

$$= \sum_{j=1}^S A_{Tj} / S$$

(217)

So, $\text{Var}(\bar{Y}_T) = \frac{1}{S^2} \bar{h}' \text{Var}(\bar{a}) \bar{h}$ (218)

By equation (13), $\text{Var}(\bar{a}) = I\sigma_e^2 + \sigma_b^2 UU'$, equation (218) can be written as

$$\text{Var}(\bar{Y}_T) = \frac{1}{S^2} \bar{h}' \bar{h} \sigma_e^2 + \frac{1}{S^2} \bar{h}' UU' \bar{h} \sigma_b^2 \quad (219)$$

where U is any design matrix related to equation (218) and

$$\bar{h}' \bar{h} = S$$

$$\bar{h}' UU' \bar{h} = S \quad (220)$$

That means

$$\begin{aligned} \text{Var}(\bar{Y}_T) &= \sigma_e^2 / S + \sigma_b^2 / S \\ &= \sigma_e^2 \left(1 + \sigma_b^2 / \sigma_e^2 \right) / S \end{aligned} \quad (221)$$

and with $\gamma \equiv \sigma_b^2 / \sigma_e^2$, then

$$\text{Var}(\bar{Y}_T) = \sigma_e^2(1 + \gamma)/S \quad (222)$$

3.5.1.2 Definition of the Variance-Reduction Ratio R

In order to compare the variance $\text{Var}(\hat{\alpha}_T)$ of the rotation sample current-year estimator $\hat{\alpha}_T$ to the variance $\text{Var}(\bar{Y}_T)$ of the only-based current-year estimator \bar{Y}_T , the variance-reduction ratio R is introduced as follows.

$$R \equiv \text{Var}(\hat{\alpha}_T) / \text{Var}(\bar{Y}_T)$$

That is,

$$R \equiv \text{Var}(\hat{\alpha}_T) \left\{ (1 + \gamma) \sigma_e^2 / S \right\}^{-1} \quad (223)$$

This ratio R, then, represents the gain of precision achievable by the rotation sample design estimation; and, R_i is the variance reduction ratio for the rotation pattern according to any figure i, $i = 1, 2, \dots, 12$.

3.5.1.3 Analytic Formulae for R

Based on the formulae of $\text{Var} \hat{\alpha}_T$ given in section 3.5, the formulae shown as equation (223) provides the following various analytic formulae for R according to different specified rotation patterns which are noted for figures 5-9 through 5-18 as follows.

a. Figure 3-1. ($S = 2$, $r = 2$; $T = 3$, [2 retained 1])

$$R_1 = \frac{\sigma_e^2(1 + 2\hat{\gamma})(2 + \hat{\gamma})/2(2 + 3\hat{\gamma})}{\sigma_e^2(1 + \hat{\gamma})/2}$$

by equation (52).

That is,

$$R_1 = (1 + 2\hat{\gamma})(2 + \hat{\gamma}) / (1 + \hat{\gamma})(2 + 3\hat{\gamma}) \quad (224)$$

b. Figure 3-2. ($S = 3, r = 2; T = 4, [3 \text{ retained in } 2]$)

$$R_2 = \frac{\sigma_e^2(1 + 3\hat{\gamma})(3 + 2\hat{\gamma})/3(3 + 8\hat{\gamma})}{\sigma_e^2(1 + \hat{\gamma})/3}$$

by equation (74).

That is,

$$R_2 = (1 + 3\hat{\gamma})(3 + 2\hat{\gamma})/(1 + \hat{\gamma})(3 + 8\hat{\gamma}) \quad (225)$$

c. Figure 3-3. ($S = 2, r = 3; T = 4, [2 \text{ retained } 1]$).

$$R_3 = \frac{\sigma_e^2[1 + 1/(1 + 2\hat{\gamma}) - 2/(1 + \hat{\gamma})](1 + 2\hat{\gamma})/8}{\sigma_e^2(1 + \hat{\gamma})/2}$$

by equation (79).

That is,

$$R_3 = [1 + 1/(1 + 2\hat{\gamma}) - 2/(1 + \hat{\gamma})](1 + 2\hat{\gamma})/4(1 + \hat{\gamma}) \quad (226)$$

d. Figure 3-4. ($S = 3, r = 3; T = 5, [3 \text{ retained } 2]$).

$$R_4 = \frac{\sigma_e^2[1/3 + .618/(3 + 8.618\hat{\gamma}) - 1.618/(3 + 6.382\hat{\gamma})](1 + 3\hat{\gamma})/5}{\sigma_e^2(1 + \hat{\gamma})/3}$$

by equation (96).

That is,

$$R_4 = 3[1/3 + .618/(3 + 8.618\hat{\gamma}) - 1.618/(3 + 6.382\hat{\gamma})](1 + 3\hat{\gamma})/5(1 + \hat{\gamma})$$

(227)

e. Figure 3-10. ($S = 2, r = \infty; T = 2, [2 \text{ retained } 1]$).

By equation (107),

$$R_{5a} = \frac{2(1 + 2\hat{\gamma})[(1 + 2\hat{\gamma}) + (1 + \hat{\gamma})^2]}{[(1 + 2\hat{\gamma}) + (1 + \hat{\gamma})^2] - \hat{\gamma}^2(1 + \hat{\gamma})^2} \quad (228)$$

f. Figure 3-11. ($S = 2, r = \infty; T = 3, [2 \text{ retained } 1]$).

By equation (123),

$$R_{5b} = \frac{(1 + 2\hat{\gamma}) \{2[(1 + 2\hat{\gamma}) + (1 + \hat{\gamma})^2] - \hat{\gamma}^2\}}{[(1 + 2\hat{\gamma}) + (1 + \hat{\gamma})^2]\{[(1 + 2\hat{\gamma}) + (1 + \hat{\gamma})^2] - \hat{\gamma}^2\}} \quad (229)$$

g. Figure 3-5. ($S = 2, r = \infty; T = 4, [2 \text{ retained } 1]$)

By equation (129),

$$R_5 = \frac{2(4f^{00}h - f^{002}g - 2g^{020}h)}{4f^{020}h - f^{0202}g - 4fg^{020}h + g^{04}} \quad (230)$$

where

$$f^0 = [(1 + 2\hat{\gamma}) + (1 + \hat{\gamma})^2]/(1 + 2\hat{\gamma})$$

$$g^0 = -\hat{\gamma}(1 + \hat{\gamma})/(1 + 2\hat{\gamma})$$

$$h^0 = (1 + \hat{\gamma})^2/(1 + 2\hat{\gamma})$$

h. Figure 3-12. ($S = 3, r = \infty; T = 3 [3 \text{ retained } 2]$).

By equation (142),

$$R_{6a} = \frac{3(AC^0 - B^02)}{(A - D)[(A + D)C^0 - 2B^02]} \quad (231)$$

i. Figure 3-6. ($S = 3, r = \infty; T = 4, [3 \text{ retained } 2]$).

By equation (157),

$$R_6 = 3 \begin{pmatrix} 0 & 0 & 0 & 0 \\ A & -B & M^{-1} & B \end{pmatrix}^{-1} \quad (232)$$

j. Figure 3-13. ($S = 4, r = \infty; T = 3, [4 \text{ retained } 3]$)

By equation (174),

$$R_{7a} = \frac{4(\overset{00}{F}\overset{00}{H} - \overset{00}{G}^2)}{(\overset{0}{F} - 2\overset{0}{D})[(\overset{0}{F} + 2\overset{0}{D})\overset{0}{H} - 2\overset{00}{G}^2]} \quad (233)$$

where $\overset{0}{F}$, $\overset{0}{G}$, $\overset{0}{H}$, and $\overset{0}{D}$ are defined in equation (175) through (178).

k. Figure 3-7. ($S = 4, r = \infty; T = 4, [4 \text{ retained } 3]$).

By equation (192),

$$R_7 = 4 \left\{ \overset{0}{J} - \overset{0}{\underline{y}}, \overset{0-1}{N} \overset{0}{\underline{y}} \right\} - 1 \quad (234)$$

where $\overset{0}{J}$, $\overset{0}{\underline{y}}$, $\overset{0}{N}$ are defined in equations (194) through (201).

l. Figure 3-14. ($S = 4, r = 3; T = 6, [4 \text{ retained } 3]$).

By equation (203),

$$R_8 = \frac{2(1 + 4\hat{\gamma})}{3(1 + \hat{\gamma})} \left(\frac{1}{4} + \frac{1}{4 + 16\hat{\gamma}} - \frac{\sqrt{3}}{4 + 14\hat{\gamma} - \sqrt{3}\hat{\gamma}} - \frac{\sqrt{3}}{4 + 10\hat{\gamma} + 5\sqrt{3}\hat{\gamma}} \right) \quad (235)$$

m. Figure 3-15. ($S = 4, r = 4; T = 7, [4 \text{ retained } 3]$).

By equation (206),

$$R_9 = \frac{4(1 + 4\hat{\gamma})}{7(1 + \hat{\gamma})} \left(\frac{1}{4} + \frac{1.470}{4 + 15.357\hat{\gamma}} - \frac{1.802}{4 + 15.602\hat{\gamma}} - \frac{0.445}{4 + 10.951\hat{\gamma}} \right) \quad (236)$$

n. Figure 3-16. ($S = 4, r = 2; T = 3, [4 \text{ retained } 2]$).

By equation (209),

$$R_{10} = (1 + 2\hat{\gamma})(2 + \hat{\gamma}) / (1 + \hat{\gamma})(2 + 3\hat{\gamma}) = R_1 \quad (237)$$

o. Figure 3-17. ($S = 4, r = 3; T = 4, [4 \text{ retained } 2]$).

By equation (212),

$$R_{11} = \frac{1 + 2\hat{\gamma}}{4(1 + \hat{\gamma})} \left(1 + \frac{1}{1 + 2\hat{\gamma}} - \frac{2}{1 + \hat{\gamma}} \right) = R_3 \quad (238)$$

p. Figure 3-18. ($S = 4$, $r = \infty$; $T = 3$, [4 retained 2]).

By equation (215),

$$R_{12} = (2f^{\overline{00}}h^{\overline{00}} - g^{\overline{02}})/f^{\overline{00}}(fh^{\overline{00}} - g^{\overline{02}}) \quad (239)$$

where $f^{\overline{00}}$, $h^{\overline{00}}$, and $g^{\overline{02}}$ are defined in equation (230).

3.5.2 VALUES OF $\text{Var}(\hat{\alpha}_T)$ AND R FOR A SPECIFIED VALUE OF $\hat{\gamma}$

Table 3-1 provides a summary of the rotation pattern figures given in section 3 of this document. In addition, tables 3-2, 3-3, 3-4, and 3-5 show the values of $\text{Var}(\hat{\alpha}_T)$ and R corresponding to the first seven rotation designs for two specified values of $\hat{\gamma}$, $\hat{\gamma} = 2.01$ and $\hat{\gamma} = 3.38$.

Based on a limited data bank, the estimation of γ was performed in two cases as follows.

a. Case 1

A 4-year set of data in which all acquisitions for all years were used; for this case, the estimate was $\hat{\gamma} = 2.01$.

b. Case 2

A situation such as in Case 1, but one in which the first acquisition estimates in the fourth year were used; for this case, the estimate was $\hat{\gamma} = 3.38$.

At this point, it is necessary to note that the estimate of γ computed is not strictly a ratio of $\hat{\sigma}_b^2$ and $\hat{\sigma}_e^2$ which is computed for a single stratum. The data are not adequate for such an analysis. Instead, the available data for all strata were pooled and a simulation of σ_b^2 and σ_e^2 was computed by fitting the model

$$y_{tsc} = \alpha_t + \gamma_c + (\alpha\gamma)_{tc} + b_s + e_{tsc} \quad (240)$$

where the subscript c denotes the stratum (CRD) of the segment, the α_t , γ_c , and $(\alpha\gamma)_{tc}$ are year, stratum (CRD), and interaction constants, respectively, and the b_s , and e_{tsc} are, as before, segment and error variables.

TABLE 3-1.- SUMMARY OF ROTATION PATTERN FIGURES

Figure	Rotation pattern	Analytic formulae of Var $\hat{\alpha}_T$ and R																																				
<p>Fig. 3-1</p> <p>Year number</p> <table><tr><td>s \ t</td><td>1</td><td>2</td><td>3</td></tr><tr><td>1</td><td>x</td><td></td><td>x</td></tr><tr><td>2</td><td>x</td><td>x</td><td></td></tr><tr><td>3</td><td></td><td>x</td><td>x</td></tr></table> <p>Segment number</p>	s \ t	1	2	3	1	x		x	2	x	x		3		x	x	(S = 2, r = 2; T = 3, [2 retained 1])	$\text{Var } \hat{\alpha}_T = \frac{(1 + 2\hat{\gamma})(2 + \hat{\gamma})}{2(2 + \hat{\gamma})} \sigma_e^2$ $\hat{\gamma} \equiv \hat{\sigma}_b^2 / \hat{\sigma}_e^2, \text{ as estimated from pilot data}$ $R = \frac{(1 + 2\hat{\gamma})(2 + \hat{\gamma})}{(1 + \hat{\gamma})(2 + 3\hat{\gamma})}$																				
s \ t	1	2	3																																			
1	x		x																																			
2	x	x																																				
3		x	x																																			
<p>Fig. 3-2</p> <p>Year number</p> <table><tr><td>s \ t</td><td>1</td><td>2</td><td>3</td><td>4</td></tr><tr><td>1</td><td>x</td><td></td><td>x</td><td>x</td></tr><tr><td>2</td><td>x</td><td>x</td><td></td><td>x</td></tr><tr><td>3</td><td>x</td><td>x</td><td>x</td><td></td></tr><tr><td>4</td><td></td><td>x</td><td>x</td><td>x</td></tr></table> <p>Segment number</p>	s \ t	1	2	3	4	1	x		x	x	2	x	x		x	3	x	x	x		4		x	x	x	(S = 3, r = 2; T = 4, [3 retained 2])	$\text{Var } \hat{\alpha}_T = \frac{(1 + 3\hat{\gamma})(3 + 2\hat{\gamma})}{3(3 + 8\hat{\gamma})} \sigma_e^2$ $R = \frac{(1 + 3\hat{\gamma})(3 + 2\hat{\gamma})}{(1 + \hat{\gamma})(3 + 8\hat{\gamma})}$											
s \ t	1	2	3	4																																		
1	x		x	x																																		
2	x	x		x																																		
3	x	x	x																																			
4		x	x	x																																		
<p>Fig. 3-3</p> <p>Year number</p> <table><tr><td>s \ t</td><td>1</td><td>2</td><td>3</td><td>4</td></tr><tr><td>1</td><td>x</td><td></td><td></td><td>x</td></tr><tr><td>2</td><td>x</td><td>x</td><td></td><td></td></tr><tr><td>3</td><td></td><td>x</td><td>x</td><td></td></tr><tr><td>4</td><td></td><td></td><td>x</td><td>x</td></tr></table> <p>Segment number</p>	s \ t	1	2	3	4	1	x			x	2	x	x			3		x	x		4			x	x	(S = 2, r = 3; T = 4, [2 retained 1])	$\text{Var } \hat{\alpha}_T = \frac{1 + 2\hat{\gamma}}{8} \left(1 + \frac{1}{1 + 2\hat{\gamma}} - \frac{2}{1 + \hat{\gamma}} \right) \sigma_e^2$ $R = \frac{(1 + 2\hat{\gamma})}{4(1 + \hat{\gamma})} \left(1 + \frac{1}{1 + 2\hat{\gamma}} - \frac{2}{1 + \hat{\gamma}} \right)$											
s \ t	1	2	3	4																																		
1	x			x																																		
2	x	x																																				
3		x	x																																			
4			x	x																																		
<p>Fig. 3-4</p> <p>Year number</p> <table><tr><td>s \ t</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td></tr><tr><td>1</td><td>x</td><td></td><td></td><td>x</td><td>x</td></tr><tr><td>2</td><td>x</td><td>x</td><td></td><td></td><td>x</td></tr><tr><td>3</td><td>x</td><td>x</td><td>x</td><td></td><td></td></tr><tr><td>4</td><td></td><td>x</td><td>x</td><td>x</td><td></td></tr><tr><td>5</td><td></td><td></td><td>x</td><td>x</td><td>x</td></tr></table> <p>Segment number</p>	s \ t	1	2	3	4	5	1	x			x	x	2	x	x			x	3	x	x	x			4		x	x	x		5			x	x	x	(S = 3, r = 3; T = 5, [3 retained 2])	$\text{Var } \hat{\alpha}_T = \frac{1 + 3\hat{\gamma}}{5} \left(\frac{1}{3} + \frac{0.618}{3 + 8.618\hat{\gamma}} - \frac{1.618}{3 + 6.382\hat{\gamma}} \right) \sigma_e^2$ $R = \frac{3(1 + 3\hat{\gamma})}{5(1 + \hat{\gamma})} \left(\frac{1}{3} + \frac{0.618}{3 + 8.618\hat{\gamma}} - \frac{1.618}{3 + 6.382\hat{\gamma}} \right)$
s \ t	1	2	3	4	5																																	
1	x			x	x																																	
2	x	x			x																																	
3	x	x	x																																			
4		x	x	x																																		
5			x	x	x																																	

TABLE 3-1.- Continued.

Figure	Rotation patterns	Analytic formulae of Var $\hat{\alpha}_T$ and R																												
<p>Fig. 3-11</p> <table><tr><td colspan="4">Year number</td></tr><tr><td>1</td><td>1</td><td>2</td><td>3</td></tr><tr><td>2</td><td></td><td></td><td></td></tr><tr><td>3</td><td></td><td></td><td></td></tr><tr><td>4</td><td></td><td></td><td></td></tr></table>	Year number				1	1	2	3	2				3				4				(S = 2, r = ∞; T = 3, [2 retained 1])	$\text{Var } \hat{\alpha}_T = \frac{(1 + \hat{\gamma})(1 + 2\hat{\gamma})(2\hat{d} - \hat{\gamma}^2)}{2\hat{d}(\hat{d} - \hat{\gamma}^2)} \sigma_e^2$ $R = \frac{(1 + 2\hat{\gamma})(2\hat{d} - \hat{\gamma}^2)}{2\hat{d}(\hat{d} - \hat{\gamma}^2)}$								
Year number																														
1	1	2	3																											
2																														
3																														
4																														
<p>Fig. 3-5</p> <table><tr><td colspan="4">Year number</td></tr><tr><td>1</td><td>1</td><td>2</td><td>3</td></tr><tr><td>2</td><td></td><td></td><td></td></tr><tr><td>3</td><td></td><td></td><td></td></tr><tr><td>4</td><td></td><td></td><td></td></tr><tr><td>5</td><td></td><td></td><td></td></tr></table>	Year number				1	1	2	3	2				3				4				5				(S = 2, r = ∞; T = 4, [2 retained 1])	$\text{Var } \hat{\alpha}_T = \frac{(1 + \hat{\gamma})(4fh - fg^2 - 2g^2h)}{4f^2h - fg^2 - 4fg^2h + g^4} \sigma_e^2$ <p>where</p> $f \equiv \hat{d}/(1 + 2\hat{\gamma})$ $g \equiv -\hat{\gamma}(1 + \hat{\gamma})/(1 + 2\hat{\gamma})$ $h \equiv (1 + \hat{\gamma})^2/(1 + 2\hat{\gamma})$ $R = \frac{2(4fh - fg^2 - 2g^2h)}{4f^2h - fg^2 - 4fg^2h + g^4}$				
Year number																														
1	1	2	3																											
2																														
3																														
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5																														
<p>Fig. 3-13</p> <table><tr><td colspan="4">Year number</td></tr><tr><td>1</td><td>1</td><td>2</td><td>3</td></tr><tr><td>2</td><td></td><td></td><td></td></tr><tr><td>3</td><td></td><td></td><td></td></tr><tr><td>4</td><td></td><td></td><td></td></tr><tr><td>5</td><td></td><td></td><td></td></tr><tr><td>6</td><td></td><td></td><td></td></tr></table>	Year number				1	1	2	3	2				3				4				5				6				(S = 4, r = ∞; T = 3, [4 retained 3])	$\text{Var } \hat{\alpha}_T = \frac{(1 + \hat{\gamma})(FH - G^2)}{(F - 2\hat{\gamma})(F + 2\hat{\gamma})H - 2G^2}$ <p>where</p> $(1 + 2\hat{\gamma})(1 + 3\hat{\gamma})F = (1 + 2\hat{\gamma})(1 + 3\hat{\gamma}) + (1 + \hat{\gamma})^2(1 + 3\hat{\gamma}) + 2(1 + \hat{\gamma})(1 + 2\hat{\gamma})^2$ $(1 + 2\hat{\gamma})(1 + 3\hat{\gamma})G = -\hat{\gamma}(1 + 2\hat{\gamma})[2(1 + 2\hat{\gamma}) + (1 + 3\hat{\gamma})]$ $(1 + 2\hat{\gamma})(1 + 3\hat{\gamma})H = 2(1 + \hat{\gamma})[(1 + \hat{\gamma})(1 + 3\hat{\gamma}) + (1 + 2\hat{\gamma})^2]$ $R = \frac{FH - G^2}{(F - 2\hat{\gamma})(F + 2\hat{\gamma})H - 2G^2}$
Year number																														
1	1	2	3																											
2																														
3																														
4																														
5																														
6																														

TABLE 3-1.- Continued.

Figure	Rotation pattern	Analytic formulae of Var $\hat{\alpha}_T$ and R																																			
<p>Fig. 3-12</p> <table><tr><td>Year number</td><td>1</td><td>2</td><td>3</td></tr><tr><td>1</td><td>x</td><td></td><td></td></tr><tr><td>2</td><td></td><td>x</td><td></td></tr><tr><td>3</td><td></td><td></td><td>x</td></tr><tr><td>4</td><td></td><td></td><td></td></tr><tr><td>5</td><td></td><td></td><td></td></tr></table> <p>Segment number</p>	Year number	1	2	3	1	x			2		x		3			x	4				5				$(S = 3, r = \infty; T = 3, [3 \text{ retained } 2])$	$\text{Var } \hat{\alpha}_T = \frac{(1 + \tilde{\gamma})(\frac{00}{0} \frac{00}{0} - \frac{02}{0} \frac{02}{0})}{(A - \tilde{D})[(A + \tilde{D})C - 2\tilde{B}^2]} \sigma_e^2$ <p>where</p> $(1 + 2\tilde{\gamma})(1 + 3\tilde{\gamma})A^0 = (1 + 2\tilde{\gamma})(1 + 3\tilde{\gamma}) + (1 + \tilde{\gamma})^2(1 + 3\tilde{\gamma}) + (1 + \tilde{\gamma})(1 + 2\tilde{\gamma})^2$ $(1 + 2\tilde{\gamma})(1 + 3\tilde{\gamma})B^0 = -\tilde{\gamma}(1 + \tilde{\gamma})[(1 + 2\tilde{\gamma}) + (1 + 3\tilde{\gamma})]$ $(1 + 2\tilde{\gamma})(1 + 3\tilde{\gamma})C^0 = (1 + \tilde{\gamma})[2(1 + \tilde{\gamma})(1 + 3\tilde{\gamma}) + (1 + 2\tilde{\gamma})^2]$ $(1 + 2\tilde{\gamma})(1 + 3\tilde{\gamma})D^0 = -\tilde{\gamma}(1 + \tilde{\gamma})(1 + 2\tilde{\gamma})$ $R = \frac{3(\frac{00}{0} \frac{00}{0} - \frac{02}{0} \frac{02}{0})}{(A - \tilde{D})[(A + \tilde{D})C - 2\tilde{B}^2]}$											
Year number	1	2	3																																		
1	x																																				
2		x																																			
3			x																																		
4																																					
5																																					
<p>Fig. 3-6</p> <table><tr><td>Year number</td><td>1</td><td>2</td><td>3</td><td>4</td></tr><tr><td>1</td><td>x</td><td></td><td></td><td></td></tr><tr><td>2</td><td></td><td>x</td><td></td><td></td></tr><tr><td>3</td><td></td><td></td><td>x</td><td></td></tr><tr><td>4</td><td></td><td></td><td></td><td>x</td></tr><tr><td>5</td><td></td><td></td><td></td><td></td></tr><tr><td>6</td><td></td><td></td><td></td><td></td></tr></table> <p>Segment number</p>	Year number	1	2	3	4	1	x				2		x			3			x		4				x	5					6					$(S = 3, r = \infty; T = 4, [3 \text{ retained } 2])$	$\text{Var } \hat{\alpha}_T = (1 + \tilde{\gamma}) \left\{ A^0 - \frac{0}{\tilde{D}} \cdot \frac{0-1}{\tilde{D}} \right\}^{-1} \sigma_e^2$ <p>where</p> $(1 + 2\tilde{\gamma})(1 + 3\tilde{\gamma})E^0 = (1 + \tilde{\gamma})[(1 + \tilde{\gamma})(1 + 3\tilde{\gamma}) + 2(1 + 2\tilde{\gamma})^2]$ $\frac{0}{\tilde{D}} \cdot \frac{0}{\tilde{D}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ A & R & 0 & 0 \\ 0 & 0 & E & 2\tilde{D} \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\frac{0}{\tilde{D}} \cdot \frac{0}{\tilde{D}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $R = 3 \left\{ A^0 - \frac{0}{\tilde{D}} \cdot \frac{0-1}{\tilde{D}} \right\}^{-1}$
Year number	1	2	3	4																																	
1	x																																				
2		x																																			
3			x																																		
4				x																																	
5																																					
6																																					
<p>Fig. 3-10</p> <table><tr><td>Year number</td><td>1</td><td>2</td></tr><tr><td>1</td><td>x</td><td></td></tr><tr><td>2</td><td></td><td>x</td></tr><tr><td>3</td><td></td><td></td></tr></table> <p>Segment number</p>	Year number	1	2	1	x		2		x	3			$(S = 2, r = \infty; T = 2, [2 \text{ retained } 1])$	$\text{Var } \hat{\alpha}_T = \frac{(1 + \tilde{\gamma})(1 + 2\tilde{\gamma})\tilde{d}^2}{\tilde{d}^2 - \tilde{\gamma}^2(1 + \tilde{\gamma})^2} \sigma_e^2$ <p>where</p> $\tilde{d} \equiv (1 + 2\tilde{\gamma}) + (1 + \tilde{\gamma})^2$ $R = \frac{2(1 + 2\tilde{\gamma})\tilde{d}}{\tilde{d}^2 - \tilde{\gamma}^2(1 + \tilde{\gamma})^2}$																							
Year number	1	2																																			
1	x																																				
2		x																																			
3																																					

TABLE 3-1.- Concluded.

Figure	Rotation pattern	Analytic formulae of Var $\hat{\alpha}_T$ and R																																								
<div>Fig. 3-7</div> <div>Segment number</div> <table><tr><th>Year number</th><th>1</th><th>2</th><th>3</th><th>4</th></tr><tr><td>1</td><td>x</td><td></td><td></td><td></td></tr><tr><td>2</td><td>x</td><td>x</td><td></td><td></td></tr><tr><td>3</td><td>x</td><td>x</td><td>x</td><td></td></tr><tr><td>4</td><td>x</td><td>x</td><td>x</td><td>x</td></tr><tr><td>5</td><td></td><td>x</td><td>x</td><td>x</td></tr><tr><td>6</td><td></td><td></td><td>x</td><td>x</td></tr><tr><td>7</td><td></td><td></td><td></td><td>x</td></tr></table>	Year number	1	2	3	4	1	x				2	x	x			3	x	x	x		4	x	x	x	x	5		x	x	x	6			x	x	7				x	$(s = 4, r = \infty; T = 4, [4 \text{ retained } 3])$	<div>Analytic formulae of Var $\hat{\alpha}_T$ and R</div> <div>$\text{Var } \hat{\alpha}_T = (1 + \hat{\gamma}) \left\{ \hat{J} - \frac{0}{N} \frac{0-1}{\hat{v}} \frac{0-1}{\hat{v}} \frac{2}{\sigma_e^2} \right\}$</div> <div>where</div> <div>$\hat{J} = 1 + \frac{(1 + \hat{\gamma})^2}{1 + 2\hat{\gamma}} + \frac{(1 + \hat{\gamma})}{1 + 3\hat{\gamma}} + \frac{(1 + \hat{\gamma})(1 + 3\hat{\gamma})}{1 + 4\hat{\gamma}}$</div> <div>$\hat{K} = \frac{-\hat{\gamma}(1 + \hat{\gamma})}{1 + 2\hat{\gamma}} - \frac{\hat{\gamma}(1 + \hat{\gamma})}{1 + 3\hat{\gamma}} - \frac{\hat{\gamma}(1 + \hat{\gamma})}{1 + 4\hat{\gamma}}$</div> <div>$\hat{R} = \frac{-\hat{\gamma}(1 + \hat{\gamma})}{1 + 4\hat{\gamma}}$</div> <div>$\hat{L} = \frac{-\hat{\gamma}(1 + \hat{\gamma})}{1 + 3\hat{\gamma}} - \frac{\hat{\gamma}(1 + \hat{\gamma})}{1 + 4\hat{\gamma}}$</div> <div>$\hat{Q} = \frac{-2\hat{\gamma}(1 + \hat{\gamma})}{1 + 3\hat{\gamma}} - \frac{\hat{\gamma}(1 + \hat{\gamma})}{1 + 4\hat{\gamma}}$</div> <div>$\hat{P} = \frac{(1 + \hat{\gamma})^2}{1 + 2\hat{\gamma}} + \frac{2(1 + \hat{\gamma})(1 + 2\hat{\gamma})}{1 + 3\hat{\gamma}} + \frac{(1 + \hat{\gamma})(1 + 3\hat{\gamma})}{1 + 4\hat{\gamma}}$</div> <div>$\hat{N} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \hat{J} & \hat{K} & \hat{L} & \hat{Q} \\ 0 & 0 & \hat{P} & 0 \\ 0 & 0 & 0 & \hat{P} \end{pmatrix}$</div> <div>$\hat{v} = \begin{pmatrix} 0 & 0 & 0 \\ \hat{R} & \hat{L} & \hat{K} \end{pmatrix}$</div> <div>$R = 4 \left\{ \hat{J} - \frac{0}{N} \frac{0-1}{\hat{v}} \frac{0}{\hat{v}} \right\} - 1$</div>
Year number	1	2	3	4																																						
1	x																																									
2	x	x																																								
3	x	x	x																																							
4	x	x	x	x																																						
5		x	x	x																																						
6			x	x																																						
7				x																																						

TABLE 3-2.- VALUES OF $\sigma_e^{-2} \text{VAR}(\hat{\alpha}_T)$ FOR $\hat{\gamma} = 2.01$

No. of years in rotation, T	Values for figures with return				Values for figures without return		
	3-1	3-2	3-3	3-4	3-5	3-6	3-7
2	1.32	0.90	1.32	0.90	1.32	0.90	0.69
3	1.25	.87	1.29	.83	1.29	.83	.64
4		.86	1.17	.80	1.285	.82	.61
5				.78	1.284	.81	.59

TABLE 3-3.- VALUES OF $\sigma_e^{-2} \text{VAR}(\hat{\alpha}_T)$ FOR $\hat{\gamma} = 3.38$

No. of years in rotation, T	Values for figures with return				Values for figures without return		
	3-1	3-2	3-3	3-4	3-5	3-6	3-7
2	1.81	1.25	1.81	1.25	1.81	1.25	0.97
3	1.72	1.22	1.73	1.13	1.73	1.13	.88
4		1.21	1.54	1.0826	1.71	1.0833	.817
5				1.056	1.70	1.060	.787

TABLE 3-4 VALUES OF R FOR $\hat{\gamma} = 2.01$

No. of years in rotation T	Values for figures with return				Values for figures without return		
	3-1	3-2	3-3	3-4	3-5	3-6	3-7
2	0.877	0.897	0.877	0.897	0.877	0.897	0.917
3	.831	.867	.857	.827	.856	.827	.850
4		.857	.777	.797	.854	.817	.811
5				.777	.853	.807	.784

TABLE 3-5.- VALUES OF R FOR $\hat{\gamma} = 3.38$

No. of years in rotation, T	Values for figures with return				Values for figures without return		
	3-1	3-2	3-3	3-4	3-5	3-6	3-7
2	0.826	0.856	0.826	0.856	0.826	0.856	0.886
3	.785	.836	.790	.774	.790	.774	.804
4		.829	.703	.741	.781	.742	.746
5				.723	.776	.726	.719

In reviewing table 3-2 through 3-5, the following note is pertinent. As mentioned in section (3.5.1.2.2), in the case of rotation with return (i.e., $r < \infty$), the maximum value of T is $r + 1$ since this corresponds to a complete cycle of the rotation; T is the total number of years utilized for the rotation sample (multiyear) estimator. No entry is shown for $T = 1$ since this would be identical with the single-year estimator.

3.5.3 COMPARISONS BETWEEN ROTATION PATTERNS

Based on the values of R , instead of the values of $\text{Var } \hat{\alpha}_T$, either table 3-4 or table 3-5 will first reveal the difference between rotation patterns with return and without return. In order to have a fair comparison, the values of R in these two tables will be presented in tables 3-6 and 3-7, the lines of which show the patterns with the same number of observations.

Once the rotation patterns with return (i.e. $r < \infty$) reach their cycles, their variance reduction ratios have values which are less than those of the rotation patterns without return (i.e., $r = \infty$).

In detail, tables 3-6 and 3-7 indicate the following in the case of $\hat{\gamma} = 2.01$.

$R_1 = .431 < R_5 = .856$ for $T = 3, S = 2$, 6 observations in total
 $R_3 = .777 < R_6 = .854$ for $T = 4, S = 2$, 8 observations in total
 $R_4 = .777 < R_6 = .807$ for $T = 5, S = 3$, 15 observations in total
 $R_2 = .857 > R_6 = .817$ for $T = 4, S = 3$, 12 observations in total

In the case of $\hat{\gamma} = 3.38$, tables 3-6 and 3-7 indicate the following.

$R_1 = .785 < R_5 = .790$, for $T = 3, S = 2$, 6 observations in total
 $R_3 = .703 < R_5 = .781$, for $T = 4, S = 2$, 8 observations in total
 $R_4 = .723 < R_6 = .726$, for $T = 5, S = 3$, 15 observations in total
 $R_2 = .829 > R_6 = .742$, for $T = 4, S = 3$, 12 observations in total

TABLE 3-6.- VALUES OF R IN ROTATION PATTERNS
WITH AND WITHOUT RETURN FOR $\hat{\gamma} = 2.01$

Number of observations	Rotation pattern (figures) with return for factors -				Value for R_i	Rotation pattern (figures) without return for factors -				Value for R_i
	Rotation pattern (figures) with return for factors -					Rotation pattern (figures) without return for factors -				
	Figure i	s	r	T		Figure i	s	T		
4	3-1	2	2	2	0.877	3-5	2	2	0.877	
6	3-1	2	2	3	.831	3-5	2	3	.856	
4	3-3	2	3	2	0.877	3-5	2	2	0.877	
8	3-3	2	3	3	.857	3-5	2	3	.856	
6	3-3	2	3	4	.777	3-5	2	4	.854	
6	3-2	3	2	2	0.897	3-6	3	2	0.897	
9	3-2	3	2	3	.867	3-6	3	3	.827	
12	3-2	3	2	4	.857	3-6	3	4	.817	
6	3-4	3	3	2	0.897	3-6	3	2	0.897	
9	3-4	3	3	3	.827	3-6	3	3	.827	
12	3-4	3	3	4	.797	3-6	3	4	.817	
15	3-4	3	3	5	.777	3-6	3	5	.807	

Symbol definition:

- S = number of segments observed per year
 r = number of years elapsing before segment returns to sample
 T = current year
 R_i = variance reduction ratio given by figure i

TABLE 3-7.- VALUES OF R IN ROTATION PATTERNS
WITH AND WITHOUT RETURN FOR $\hat{\gamma} = 3.38$

Number of observations	Rotation pattern (figures) with return for factors -				Value for R_i	Rotation pattern (figures) without return for factors -			Value for R_i
	Figure i	s	r	T		Figure i	s ($r = \infty$)	T	
4	3-1	2	2	2	0.826	3-5	2	2	11.826
6	3-1	2	2	3	.785	3-5	2	3	.790
4	3-3	2	3	2	0.826	3-5	2	2	0.826
6	3-3	2	3	3	.790	3-5	2	3	.790
8	3-3	2	2	4	.703	3-5	2	4	.781
6	3-2	3	2	2	0.856	3-6	3	2	0.856
9	3-2	3	2	3	.836	3-6	3	3	.774
12	3-2	3	2	4	.829	3-6	3	4	.742
6	3-4	3	3	2	0.856	3-6	3	2	0.856
9	3-4	3	3	3	.774	3-6	3	3	.774
12	3-4	3	3	4	.741	3-6	3	4	.742
15	3-4	3	3	5	.723	3-6	3	5	.726

Symbol definitions:

S = number of segments observed per year
r = number of years elapsing before segment returns to sample
T = current year
 R_i = variance reduction ratio given by figure i

Hence, when they reach full cycle, the rotation patterns with return are better than those without return. Therefore, an optimal rotation design is one which in most situations has the rotation pattern with return and with minimum value of R.

3.5.4 THE OPTIMAL ROTATION PATTERN

The optimal rotation pattern will be a rotation pattern with return and with minimum value of R over all situations.

3.5.4.1 Values of R for Rotation Patterns with Return for $\hat{\gamma} = 2.01$ and 3.38

In the case of specified values $\hat{\gamma} = 2.01$ and $\hat{\gamma} = 3.38$, the values of R among rotation patterns with return are presented in table 3-8.

TABLE 3-8.- VALUES OF R IN ROTATION PATTERNS
WITH RETURN FOR $\hat{\gamma} = 2.01$ and 3.38

No. of years in rotation, T	R values per figure, where $\hat{\gamma} = 2.01$				R values per figure, where $\hat{\gamma} = 3.38$			
	3-1	3-2	3-3	3-4	3-1	3-2	3-3	3-4
2	0.877	0.897	0.877	0.897	0.826	0.856	0.826	0.856
3	.831	.867	.857	.827	.785	.836	.790	.774
4		.857	.777	.797		.829	.703	.741
5				.777				.723

Based on these R values, figure 3-3 (S = 2, r = 3; T = 4, [2 retained 1]) seems to be an optimal rotation pattern since it yields the minimum R (R = .777 if $\hat{\gamma} = 2.01$, R = .703 if $\hat{\gamma} = 3.38$). This indicates that, with this rotation pattern, the cost of AgRISTARS could be reduced to about three-fourths.

3.5.4.2 Values of R for Rotation Patterns with Return for Various Values of $\hat{\gamma}$

In a previous section, an optimal rotation pattern was chosen based on the values of R confined in two specified values of $\hat{\gamma}$ (2.01 and 3.38). Now the empirical optimization will take place based on values of R computed from various values of $\hat{\gamma}$. Table 3-9 presents those values.

TABLE 3-9.- VALUES OF VAR $\hat{\alpha}_T$ IN DIFFERENT ROTATION PATTERNS WITH
RETURN FOR VARIOUS VALUES OF $\hat{\gamma}$

Values of $\hat{\gamma}$	Values of Var $\hat{\alpha}_T$ for fig. 3.1 after T yrs. in rotation		Values of Var $\hat{\alpha}_T$ for fig. 3.2 after T yrs. in rotation				Values of Var $\hat{\alpha}_T$ for fig. 3.3 after T yrs. in rotation				Values of Var $\hat{\alpha}_T$ for fig. 3.4 after T yrs. in rotation				
	T = 2	T = 3	T = 2	T = 3	T = 4	T = 2	T = 3	T = 4	T = 2	T = 3	T = 4	T = 2	T = 3	T = 4	T = 5
1.00	0.933	0.900	0.629	0.614	0.606	0.933	0.929	0.875	0.629	0.604	0.586	0.629	0.604	0.586	0.575
1.50	1.126	1.077	.764	.743	.733	1.126	1.113	1.025	.764	.721	.694	.764	.721	.694	.680
1.78	1.231	1.174	.838	.814	.804	1.231	1.211	1.105	.838	.784	.753	.838	.784	.753	.738
2.00	1.313	1.250	.896	.870	.860	1.313	1.286	1.167	.896	.833	.799	.896	.833	.799	.783
2.01	1.32	1.25	.90	.87	.86	1.32	1.29	1.17	.90	.83	.80	.90	.83	.80	.78
2.50	1.494	1.421	1.026	.997	.985	1.494	1.450	1.304	1.026	.942	.902	1.026	.942	.902	.885
3.00	1.673	1.591	1.156	1.123	1.111	1.673	1.609	1.438	1.156	1.049	1.005	1.156	1.049	1.005	.986
3.38	1.81	1.72	1.25	1.22	1.21	1.81	1.73	1.54	1.25	1.13	1.083	1.25	1.13	1.083	1.056
3.50	1.849	1.760	1.284	1.249	1.237	1.849	1.762	1.569	1.284	1.155	1.107	1.284	1.155	1.107	1.087
4.00	2.024	1.929	1.411	1.375	1.362	2.024	1.912	1.700	1.411	1.260	1.209	1.411	1.260	1.209	1.188

3.5.4.3 The Optimal Rotation Pattern

The comparison of the values of R in table 3-10 indicates that the rotation patterns in figures 3-3 and 3-4 are those better than figures 3-1 and 3-2.

Between figures 3-3 and 3-4, if the values of $\hat{\gamma}$ are greater than 1.78, the rotation pattern in figure 3-3, that is, ($S = 2$, $r = 3$, $T = 4$, [2 retained 1]), is the optimal rotation pattern.

This comparison can be figured by using the illustrations shown on the following pages (figs. 3-19, 3-20, and 3-21) in which the minimum values of R shown in figure 11 are compared with those of figures 3-1, 3-2, and 3-4.

Figure 3-19 shows the comparisons among four figures, figures 3-1 through 3-4, in terms of R values. For $T = 2$, figure 3-1 is better than the others.

Figure 3-20 gives the comparisons among three figures, figures 3-2, 3-3, and 3-4, for $T = 2$, 3, and 4 years. In figure 3-20, the R values in figure 3-2 are often higher than those of figures 3-3 and 3-4. So, figure 3-2 is eliminated to provide figure 3-21 which gives the comparisons between figure 3-3 and 3-4. (In fact, figure 3-4 needs one more year, $T = 5$, in order to achieve its full cycle). For $T = 4$, except for the period around $T = 3$, most of R values of figure 3-4 are higher than those of figure 3-3. Moreover, figure 3-3 represents a full-cycle rotation pattern ($S = 2$, $r = 3$, $T = 4$, [2 retained 1]).

The length of time ($T = 4$ years) to complete the full-cycle for the rotation is shorter than that of figure 3-4 ($T = 5$ years). Therefore, figure 3-3 is more favorable than figure 3-4.

Henceforth, figure 3-3 represents the optimal rotation pattern for $T = 4$.

TABLE 3-10.-- VALUES OF R IN DIFFERENT ROTATION PATTERNS WITH RETURN FOR VARIOUS VALUES OF γ

Values of γ	Values of R for fig. 1 after T yrs. in rotation		Values of R for fig. 2 after T yrs. in rotation		Values of R for fig. 3 after T yrs. in rotation		Values of R for fig. 4 after T yrs. in rotation					
	T = 2	T = 3	T = 2	T = 3	T = 4	T = 2	T = 3	T = 4	T = 2	T = 3	T = 4	T = 5
1.00	0.933	0.900	0.943	0.920	0.909	0.933	0.929	0.875	0.943	0.906	0.878	0.862
1.50	.901	.861	.917	.891	.880	.901	.890	.820	.917	.865	.832	.816
1.78	.886	.845	.904	.879	.868	.886	.871	.795	.904	.846	.812	.796
2.00	.875	.833	.896	.870	.860	.875	.857	.778	.896	.833	.799	.783
2.01	.877	.831	.897	.867	.857	.877	.857	.777	.897	.827	.797	.777
2.50	.854	.812	.880	.855	.845	.854	.827	.745	.880	.807	.774	.758
3.00	.836	.795	.867	.843	.833	.836	.804	.719	.867	.787	.754	.739
3.38	.826	.785	.856	.836	.829	.826	.790	.703	.856	.774	.740	.726
3.50	.822	.782	.856	.833	.804	.822	.783	.698	.856	.770	.738	.725
4.00	.810	.771	.847	.825	.817	.810	.765	.680	.847	.756	.725	.713

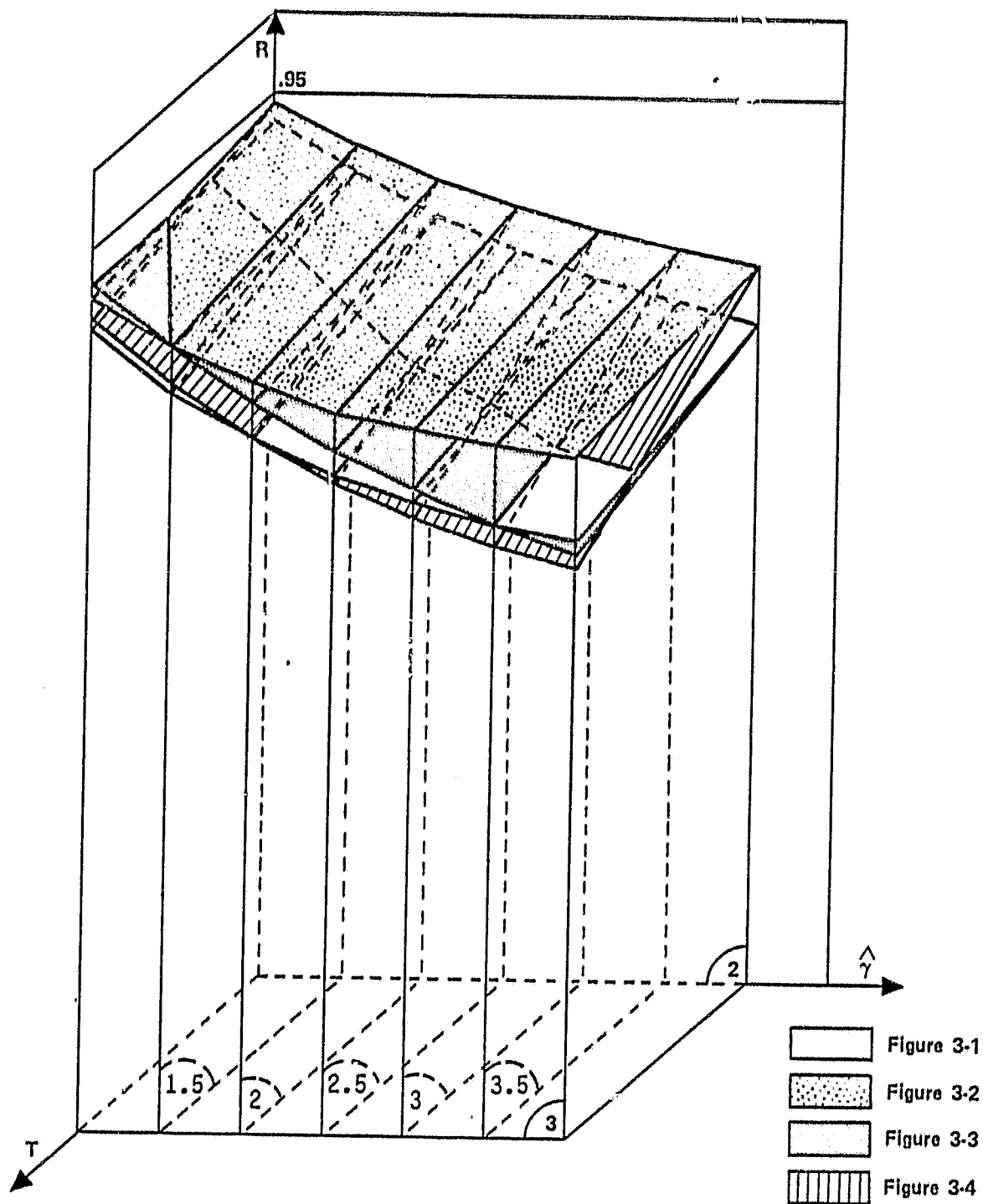


Figure 3-19.- R values in figures 3-1, 3-2, 3-3, and 3-4 by $\hat{\gamma}$ and $T = 2, 3$. For $T = 3$, figure 3-4 yields a smaller R value than do the other figures, but of all four figures, figure 3-1 shows the smallest R value. For $T = 2$, figure 3-1 shows the smallest R value.

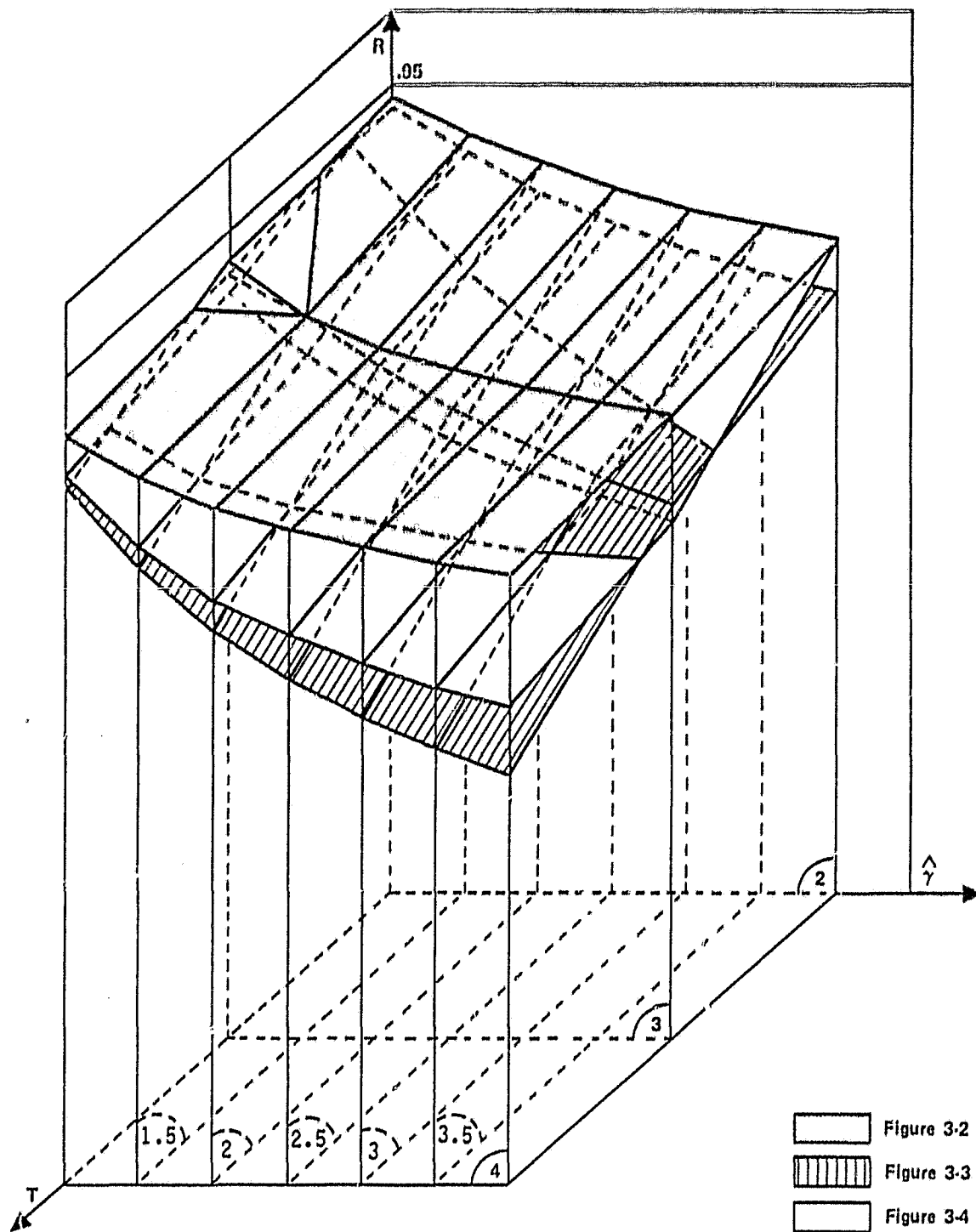


Figure 3-20.- R values in figures 3-2, 3-3, and 3-4 with respect to $\hat{\gamma}$ and T .
for $T = 4$, figure 3-3 shows a smaller R value than do the other figures.

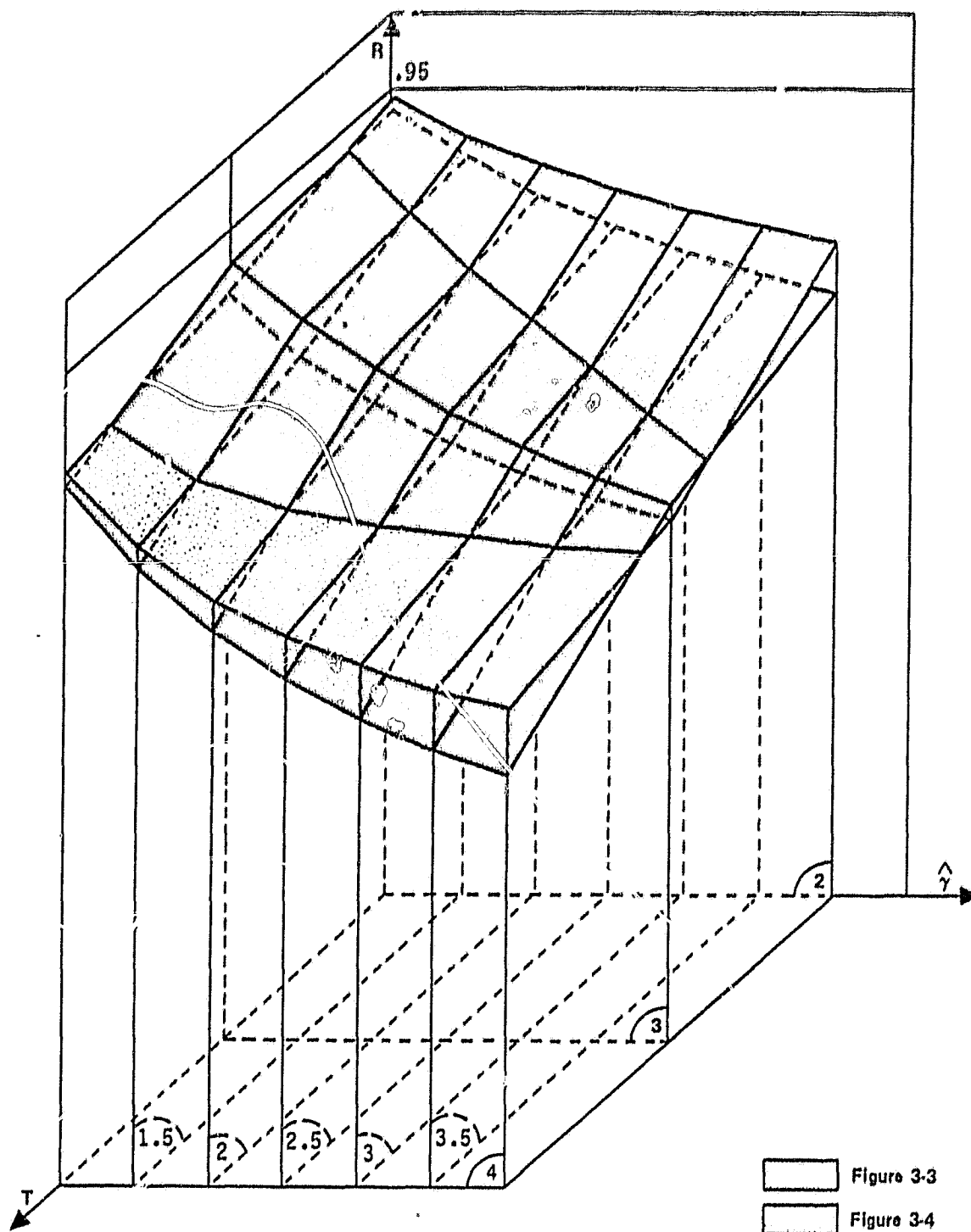


Figure 3-21.- R values in figures 3-3 and 3-4 with respect to $\hat{\gamma}$ and T. For T = 4, figure 3-3 shows a smaller R value than do the other figures.

4. CONCLUSION

4.1 THE SAMPLE DESIGN FOR FCPF

4.1.1 THE OPTIMAL ROTATION PATTERN FOR FCPF

In the case concerning the applicability of the rotation sample design for the AgRISTARS FCPF project, this study indicates that the rotation pattern ($S = 2$, $r = 3$; $T = 4$, [2 retained 1]) is the optimal rotation pattern.

It is worthwhile to recapture this optimal rotation sample design.

4.1.2 THE ROTATION SAMPLE DESIGN FOR FCPF

Average wheat acreage per sampling unit is the characteristic to be estimated by the rotation sample design for each "current" year.

The elementary units sampled are "pixels" of the earth's surface for many selected regions of some countries such as the United States, Canada, U.S.S.R., Australia, India, China, Argentina, and Brazil.

The sampling units are "segments", which are clusters of pixels. The sampling unit size is large enough that the variation of the wheat acreages of a particular segment from year to year is usually less than the variation of the wheat acreages of different segments within a particular year.

Each wheat region is stratified into L strata which are homogeneous in terms of wheat density (i.e., wheat densities vary little within a stratum but may vary considerably from stratum to stratum).

Suppose a stratum h , $h = 1, 2, \dots, L$, is comprised of N_h segments. Out of N_h , a predetermined number of segments, for example $n_h = 1, 2$, or 3 , are chosen to be in the whole sample of study.

The number n_h will be the same for each year of study, but the segments will be chosen by the rotation pattern ($S = 2$, $r = 3$; $T = 4$, [2 retained 1]), as shown in figure 3-3 which is repeated as follows.

		Year number			
Segment number	s \ t	1	2	3	4
	1	X			X
	2	X	X		
	3		X	X	
	4			X	X

(S = 2, r = 3) Rotation Pattern [2 retained 1; T = 4].

The basic model is

$$A_{ts} = \alpha_t + b_s + e_{ts} \quad (241)$$

for

t = 1, 2, 3, 4

T = 4

s = 1, 2, 3, 4

where

α_t = average true wheat acreage per segment in year t; α_t are fixed-year constants

b_s = true segment variables applicable to all years with the assumption $b_s \sim N(0, \sigma_b^2)$

e_{ts} = composite-segment-error variable of segment s in year t, with the assumption $e_{ts} \sim N(0, \sigma_e^2)$

A_{ts} = current year "direct" estimated (from satellite data) or "previous estimated" wheat acreage of segment s of stratum h in year t

In matrix form, the basic model, which is shown in equation (241), can be written as

$$\underline{a} = X\underline{\alpha} + U\underline{b} + I\underline{e} \quad (242)$$

where

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (243)$$

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (244)$$

$$\underline{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)'; \alpha_4 \text{ is to be estimated as } \hat{\alpha}_4 \quad (245)$$

$$\underline{b} = (b_1, b_2, b_3, b_4)' \sim NI_4(\underline{0}, I\sigma_b^2) \quad (246)$$

$$\underline{e} = (e_{11}, e_{41}, e_{12}, e_{22}, e_{23}, e_{33}, e_{34}, e_{44})' \sim NI_8(\underline{0}, I\sigma_e^2) \quad (247)$$

$$\underline{a} = (A_{11}, A_{41}, A_{12}, A_{22}, A_{23}, A_{33}, A_{34}, A_{44})' \quad (248)$$

Then, the estimate $\hat{\alpha}_4$ is deduced from the vector

$$\hat{\underline{\alpha}} = (X'H^{-1}X)^{-1}X'H^{-1}\underline{a} \quad (249)$$

which is the BLUE of $\underline{\alpha}$, with

$$\text{Var } \hat{\alpha}_4 = \frac{1 + 2\hat{\gamma}}{8} \left(1 + \frac{1}{1 + \hat{\gamma}} - \frac{2}{1 + \hat{\gamma}} \right) \sigma_e^2 \quad (250)$$

where

$$\hat{\gamma} = \hat{\sigma}_b^2 / \hat{\sigma}_e^2, \text{ as estimated from the pilot data} \quad (251)$$

$$H = I + \hat{\gamma}UU' \quad (252)$$

$$H^{-1} = \frac{1}{1 + 2\hat{\gamma}} \begin{bmatrix} 1 + \hat{\gamma} & -\hat{\gamma} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\hat{\gamma} & 1 + \hat{\gamma} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 + \hat{\gamma} & -\hat{\gamma} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\hat{\gamma} & 1 + \hat{\gamma} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 + \hat{\gamma} & -\hat{\gamma} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\hat{\gamma} & 1 + \hat{\gamma} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 + \hat{\gamma} & -\hat{\gamma} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\hat{\gamma} & 1 + \hat{\gamma} \end{bmatrix} \quad (253)$$

$$X'H^{-1}X = \frac{1}{1 + 2\hat{\gamma}} \text{Circ}[2(1 + \hat{\gamma}), -\hat{\gamma}, 0, -\hat{\gamma}] \quad (254)$$

Therefore, the estimate of the stratum h wheat acreage at the year $T \equiv 4$ will be

$$\hat{A}_{4h} = N_h \hat{\alpha}_4 \quad (255)$$

with the variance

$$\text{Var}(\hat{A}_{4h}) = N_h^2 \text{Var} \hat{\alpha}_4 \quad (256)$$

And, if \hat{y}_{4h} is the current-year stratum-h average wheat yield estimate, the estimate of the total wheat production in the wheat region of interest will be

$$\hat{p}_{\text{prod}_4} = \sum_{h=1}^L (\hat{A}_{4h} \hat{y}_{4h}) \quad (257)$$

From table 3-10, note that when $\hat{\gamma} = 4.00$, $R = .68$. That is, with this optimal rotation sample design, the reduced variances of the rotation (multiyear) design estimates go to 68 percent of the corresponding variances for the one year estimates.

4.2 LIMITATIONS

The first limitation, as mentioned previously, is that the target population must yield $n_h = 1, 2, \text{ or } 3$ sample segments per stratum h . This kind of target population was named Group I.

The second restraint is that the sampling unit size needs to be large enough so that the variation of wheat acreages of a particular segment from year to year is less than the variation of wheat acreages of different segments within a particular year.

Hartley et al. (ref. 4) have shown that if the estimate of σ 's (hence of γ) is consistent, a formula similar to equation (29) applies asymptotically. Therefore, the third requirement is that the estimate of γ needs to be consistent so that the formula shown as equation (250) is valid. The estimate $\hat{\gamma}$ is obtained from the pilot data.

5. REFERENCES

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APPENDIX A
THE INVERSE OF A SPECIAL MATRIX

APPENDIX A
THE INVERSE OF A SPECIAL MATRIX

For a special matrix of the form

$$(a_{ij}) = \begin{bmatrix} a & b & \cdots & b \\ b & a & \cdots & b \\ b & b & \cdots & a \end{bmatrix}_{n \times n} \quad (A-1)$$

it is easy to prove that

$$|a_{ij}| = (a - b)^{n-1} [a + (n - 1)b] \quad (A-2)$$

and that the inverse is also of the same form, but now

$$a^{ii} = \frac{a + (n - 2)b}{[a + (n - 1)b] (a - b)} \quad (A-3)$$

and

$$a^{ij} = \frac{-b}{[a + (n - 1)b] (a - b)}, \quad i \neq j$$

APPENDIX B

THE INVERSE OF A SPECIAL TRIDIAGONAL MATRIX

APPENDIX B

INVERSION OF A SPECIAL TRIDIAGONAL MATRIX

A special method which has been devised using finite difference calculus will be utilized to inverse the following special tridiagonal matrix. This matrix is of the form

$$\begin{bmatrix} a & c & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ c & b & c & 0 & & & & \cdot \\ 0 & c & b & c & & & & \cdot \\ \cdot & & \cdot & \cdot & & & & \cdot \\ \cdot & & & \cdot & c & b & c & 0 \\ \cdot & & & & 0 & c & a & c \end{bmatrix} \quad (B-1)$$

The last column of the inverse of matrix (B-1) will be denoted as

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_T \end{bmatrix} \quad (B-2)$$

The equation of the definition of the matrix inverse

$$AA^{-1} = I$$

will give a system of T equations. Not considering the first and the last equations, the $(T - 2)$ middle equations are

$$x_{i-1}c + x_i b + x_{i+1}c = 0 \quad (B-3)$$

which are

$$x_{i-1} - 2x_i + x_{i+1} = \frac{-2c - b}{c} x_i \quad (B-4)$$

denoted as

$$\Delta'' x_i = \lambda x_i \quad (B-5)$$

where

$$\lambda \equiv -(b + 2c)/c$$

A solution is

$$x_i = \rho^i \quad (B-6)$$

so, equation (B-4) becomes

$$\rho^2 - (2 + \lambda)\rho + 1 = 0 \quad (B-7)$$

which has two roots ρ_1 and ρ_2 .

Hence, a general solution will be

$$x_i = \alpha \rho_1^i + \beta \rho_2^i \quad (B-8)$$

where α and β are found from the first and last equations which are

$$\begin{cases} a(\alpha \rho_1 + \beta \rho_2) + c(\alpha \rho_1^2 + \beta \rho_2^2) = 0 \\ c(\alpha \rho_1^{n-1} + \beta \rho_2^{n-1}) + a(\alpha \rho_1^n + \beta \rho_2^n) = 1 \end{cases} \quad (B-9)$$

$$\begin{cases} c(\alpha \rho_1^{n-1} + \beta \rho_2^{n-1}) + a(\alpha \rho_1^n + \beta \rho_2^n) = 1 \end{cases} \quad (B-10)$$

With the condition

$$4\lambda + \lambda^2 > 0 \quad (B-11)$$

which is the positive discriminant of equation (B-7), the two roots ρ_1 and ρ_2 will be

$$\rho_1, \rho_2 = 1 + \frac{\lambda}{2} \pm \frac{1}{2} \sqrt{4\lambda + \lambda^2} \quad (B-12)$$

Two equations (B-9) and (B-10), hence, will give

$$\beta = \left\{ \left[c(\rho_1^{n-1} + \rho_2^{n-1}) \right] + a \left[\rho_1^n + \rho_2^n \right] \right\}^{-1} \quad (B-13)$$

$$\alpha = \rho_1 \beta$$

where

$$\zeta \equiv -\left(c\rho_2^2 + a\rho_2\right) / \left(c\rho_1^2 + a\rho_2\right) \quad (\text{B-14})$$

Therefore,

$$x_T = \alpha\rho_1^T + \beta\rho_2^T \quad (\text{B-15})$$

APPENDIX C
THE INVERSE OF A CIRCULANT MATRIX

APPENDIX C

THE INVERSE OF A CIRCULANT MATRIX

The following presentation is provided by Dr. H. G. Newton, professor, Texas A&M University.

A circulant matrix is a matrix of the form

$$A = \begin{bmatrix} a(1) & a(2) & \cdots & a(T) \\ a(T) & a(1) & \cdots & a(T-1) \\ - & - & - & - \\ a(2) & a(3) & \cdots & a(1) \end{bmatrix} \quad (C-1)$$

which is denoted as

$$A = \text{Circ}[a(1), \dots, a(T)]$$

Equation (C-1) can be written as

$$A = P\Lambda P' \quad (C-2)$$

where

$$P = (p_1, \dots, p_T)$$

$$\Lambda = \text{diag}(\lambda_1, \dots, \lambda_T)$$

with

$$\lambda_j \equiv \sum_{k=1}^T a(k) e^{-\frac{i2\pi(j-1)(k-1)}{T}} ; \quad j = 1, 2, \dots, T \quad (C-3)$$

$$T^{1/2} p_j = \begin{bmatrix} 1 \\ e^{-i2\pi(j-1)(1)/T} \\ e^{-i2\pi(j-1)(2)/T} \\ \vdots \\ e^{-i2\pi(j-1)(T-1)/T} \end{bmatrix} ; \quad j = 1, 2, \dots, T \quad (C-4)$$

Since

$$P'P = PP' = I_T$$

$$(P')^{-1} = P$$

$$P^{-1} = P'$$

thus,

$$\begin{aligned} A^{-1} &= (P \Lambda P')^{-1} \\ &= (P')^{-1} \Lambda^{-1} P^{-1} \\ &= P \Lambda^{-1} P \end{aligned} \tag{C-5}$$

And, the (j, k) element of A^{-1} will be

$$\begin{aligned} (A^{-1})_{jk} &= j^{\text{th}} \text{ row of } P \text{ times } k^{\text{th}} \text{ column of } (\Lambda^{-1}P') \\ &= \sum_{l=1}^T P_{jl} (\Lambda^{-1}P')_{lk} \\ &= \sum_{l=1}^T P_{jl} (l^{\text{th}} \text{ row of } \Lambda^{-1} \text{ times } k^{\text{th}} \text{ row of } P) \\ &= \sum_{l=1}^T P_{jl} (\Lambda^{-1})_{ll} P_{kl} \\ &= \frac{1}{T} \sum_{l=1}^T \frac{e^{-i2\pi[(j-1)(l-1) + (k-1)(l-1)]/T}}{\lambda_l} \\ &= \frac{1}{T} \sum_{l=1}^T \frac{e^{-i2\pi(l-1)(j-1 + k-1)/T}}{\lambda_l} \\ &= \frac{1}{T} \sum_{l=1}^T \frac{\exp[-i2\pi(l-1)2(j-1)/T]}{\lambda_l} \end{aligned} \tag{C-6}$$

hence

$$(A^{-1})_{jj} = \frac{1}{T} \sum_{l=1}^T \frac{e^{-i2\pi(l-1)2(j-1)/T}}{\lambda_l} \quad (C-7)$$

and

$$(A^{-1})_{11} = \frac{1}{T} \sum_{l=1}^T \frac{1}{\lambda_l} \quad (C-8)$$

$$(A^{-1})_{TT} = \frac{1}{T} \sum_{l=1}^T \frac{e^{-i2\pi(l-1)2(T-1)/T}}{\lambda_l} \quad (C-9)$$

For example,

$$A = \text{Circ}(2(1 + \gamma), -\gamma, 0, -\gamma)$$

$$\lambda_1 = 2(1 + \gamma) - \gamma + 0 - \gamma = 2$$

$$\begin{aligned} \lambda_2 &= 2(1 + \gamma) + (-\gamma) e^{-i2\pi/4} + (-\gamma) e^{-i6\pi/4} \\ &= 2(1 + \gamma) \end{aligned} \quad (C-10)$$

$$\begin{aligned} \lambda_3 &= 2(1 + \gamma) + (-\gamma) e^{-i4\pi/4} + (-\gamma) e^{-i12\pi/4} \\ &= 2 + 4\gamma \end{aligned}$$

$$\begin{aligned} \lambda_4 &= 2(1 + \gamma) + (-\gamma) e^{-i6\pi/4} + (-\gamma) e^{-i18\pi/4} \\ &= 2 + 2\gamma \end{aligned}$$

$$\begin{aligned} (A^{-1})_{44} &= \frac{1}{4} \left\{ \frac{1}{\lambda_1} + \frac{e^{-i3\pi}}{\lambda_2} + \frac{e^{-i6\pi}}{\lambda_3} + \frac{e^{-i9\pi}}{\lambda_4} \right\} \\ &= \frac{1}{4} \left\{ \frac{1}{2} - \frac{1}{2(1 + \gamma)} + \frac{1}{2 + 4\gamma} - \frac{1}{2 + 2\gamma} \right\} \\ &= \frac{1}{8} \left\{ 1 - \frac{2}{1 + \gamma} + \frac{1}{1 + 2\gamma} \right\} \end{aligned} \quad (C-11)$$